Etude à 1.15 GeV/c de la création associée $\pi^+ + p \to \Sigma^+ + K^+$ (*).

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Summary. — 78 associated production events $\pi^+ + p \to \Sigma^+ + K^+$ have been investigated by means of a propane bubble chamber (40 cm × × 20 cm × 20 cm) placed in a magnetic field of 12.7 kG. The beam used had a momentum of (1.155±0.025) GeV/c, and had the following composition: $\pi^+ : p : \mu^+ : e^+ = 100 : 10 : 8 : 1$. We have obtained following results: 1) The total $(\sigma = (0.214 \pm 0.035)$ mb) and differential cross sections obtained are in agreement with the charge independence hypothesis; 2) for $\Sigma^+ \to \pi^+ + n$ events, no significant asymmetries, up-down, right-left or forward-backward, were found; 3) the mean life of Σ^+ is $\tau = (0.75^{+0.13}_{-0.09}) \cdot 10^{-10}$ s.

1. – Introduction.

Si les réactions de création associées:

(0) $$\pi^- + p \to \Sigma^0 + K^0$$

(1) $$\pi^- + p \to \Sigma^- + K^+$$

(2) $$\pi^+ + p \to \Sigma^+ + K^+$$

(*) Les résultats préliminaires de cette étude ont fait l'objet d'une communication à la Xᵉ Conférence des Hautes Energies (Rochester, Août 1960).

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satisfont à l'indépendance de charge, les sections efficaces différentielles et, par suite, les sections efficaces totales, doivent satisfaire à l'inégalité [1]:

$$(3) \qquad\qquad (2\sigma_0)^{\frac{1}{2}} \leqslant (\sigma_-)^{\frac{1}{2}} + (\sigma_+)^{\frac{1}{2}} .$$

Les sections efficaces concernant les réactions (0) et (1) sont relativement précises puisqu'on connaît actuellement environ 300 événements de chacun des types (0) et (1) [2]. Par contre, antérieurement à notre étude, on ne connaissait que 32 événements du type (2) [3-5]. Aussi, les incertitudes sur les sections efficaces différentielles et sur la section efficace totale de la réaction (2) restaient-elles nécessairement grandes.

CRAWFORD et al. [2] ont comparé la valeur expérimentale σ_+ avec la limite inférieure σ_+^* obtenue en appliquant l'inégalité (3) aux mesures expérimentales de σ_0 et σ_-. Compte tenu des incertitudes, l'inégalité (3) était satisfaite pour $\cos \Theta > -0.5$, Θ étant l'angle dans le système du centre de masses entre le π^+ incident et le Σ^+ créé.

Par contre, pour $\cos \Theta < -0.5$, σ_+ était inférieure de 1.6 écart-type à σ_+^*, ce qui jetait un doute sur la validité de l'hypothèse de l'indépendance de charge.

C'est pourquoi il nous a paru intéressant de reprendre l'étude de la réaction $\pi^+ + p \rightarrow \Sigma^+ + K^+$.

2. – Dispositif expérimental.

Nous avons étudié cette réaction à l'aide de la chambre à bulles à propane de Saclay de dimensions intérieures $40 \text{ cm} \times 20 \text{ cm} \times 20 \text{ cm}$ [6]. Pour éviter des interactions dans les parois épaisses d'entrée et de sortie, les centres de celles-ci sont évidés et fermés par des coupoles de 8 cm de diamètre faites en acier inoxydable de 0.7 mm d'épaisseur.

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[1] J. J. SAKURAI: *Phys. Rev.*, **107**, 908 (1957); D. FELDMAN: *Phys. Rev.*, **103**, 254 (1956).

[2] F. S. CRAWFORD, R. L. DOUGLASS, M. L. GOOD, G. R. KALBFLEISCH, M. L. STEVENSON et M. K. TICHO: *Phys. Rev. Lett.*, **3**, 394 (1959); *Annual International Conference on High Energy Physics* (Genève, 1958), p. 147.

[3] J. C. VANDER VELDE, J. W. CRONIN and D. A. GLASER: *International Conference on Mesons and Recently Discovered Particles* (Padoue-Venise, 1957), I, p. 33. *Annual International Conference on High Energy Physics* (Genève, 1958), p. 147.

[4] A. R. ERWIN, J. K. KOPP et A. M. SHAPIRO: *Phys. Rev.*, **115**, 669 (1959).

[5] W. H. HANNUM, H. COURANT, E. C. FOWLER, H. L. KRAYBILL, J. SANDWEISS et J. SANFORD: *Phys. Rev.*, **118**, 577 (1960).

[6] A. ROGOZINSKI: *L'Onde Électrique* (numéro spécial consacré à Saturne), **39**, 617 (1959).

La chambre était placée dans un champ magnétique de 12 700 G et exposée, près de l'accélérateur « Saturne » à un faisceau de π^+ rendu aussi pur et aussi monocinétique que possible.

2˙1. *Le faisceau.* – Le faisceau de π^+ utilisé avait, à l'entrée du volume sensible de la chambre, une impulsion de $(1\,155 \pm 25)$ MeV/c et sa composition répondait sensiblement aux proportions ci-dessous:

$$\pi^+ : p : \mu^+ : e^+ = 100 : 10 : 8 : 1 \,.$$

La contamination en protons a été déterminée par la méthode du temps de vol; celle en μ^+ et en e^+, à l'aide d'un compteur de Čerenkov à gaz sous pression variable pour le prêt duquel nous remercions les physiciens du Laboratoire de Physique Atomique et Moléculaire du Collège de France [7].
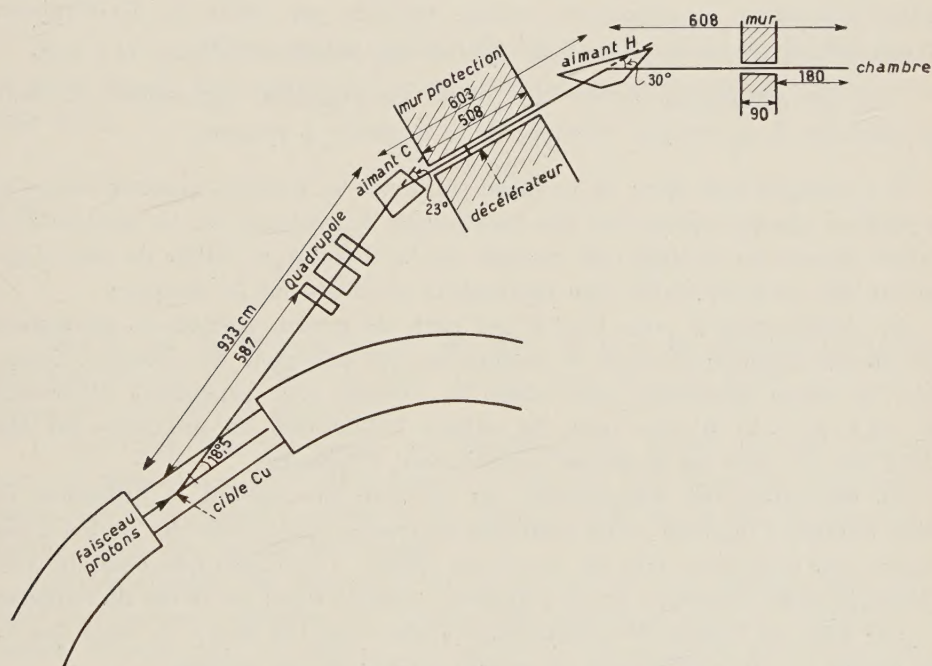

Fig. 1. – Schéma du dispositif de selection du faisceau de π^+ de 1.16 GeV/c.

Le faisceau est représenté schématiquement par la Fig. 1. Issu d'une cible en Cu, placée dans une section droite de Saturne, il est pris sous un angle

de 18°15′ par rapport au faisceau primaire de protons. La largeur apparente de la cible suivant la direction choisie est égale à 1 cm.

Avant de pénétrer dans la chambre, le faisceau traverse successivement:

1) Un ensemble de trois lentilles quadrupolaires.

2) Un premier aimant d'analyse en impulsion, où il subit une déviation moyenne de 23°. La bande d'impulsion prélevée dans le faisceau analysé (comprenant essentiellement des protons et des π^+ présents, à ce stade, dans une proportion d'environ 8 protons pour un π^+) s'étend de 1245 à 1315 MeV/c; à 3 m du centre de l'aimant, l'étalement horizontal correspondant est de 6 cm.

3) Un absorbant décélérateur, en polyéthylène, de forme particulière, décrite plus loin; ce décélérateur est disposé dans un canal de 6 cm de largeur et de 10 cm de hauteur, ménagé dans le mur de protection de Saturne.

4) Un second aimant analyseur qui effectue la séparation des composantes protonique et mésonique, rendue possible par l'effet du décélérateur, et qui fait dévier le faisceau de π^+ d'un angle moyen de 30°.

5) Un collimateur en Pb de 70 mm de diamètre, légèrement inférieur au diamètre de la coupole d'entrée de la chambre à propane.

Les réglages sont faits de la façon suivante: les lentilles quadrupolaires et le premier aimant produisent une focalisation horizontale sur le décélérateur, l'autre aimant en produit une seconde sur la chambre à bulles; de plus, l'ensemble des aimants réalise une focalisation verticale sur la chambre.

Le décélérateur a pour but, d'une part, de rendre possible la séparation des composantes protonique et mésonique qui émergent du premier aimant avec la même impulsion, mais dont les vitesses sont cependant différentes ($\beta_p \sim 0.8$, $\beta_{\pi^+} \sim 1$), d'autre part, de réduire l'étalement en impulsion du faisceau des π^+ sans en diminuer sensiblement l'intensité.

La séparation des composantes est obtenue en utilisant la différence de leurs vitesses à laquelle correspond une différence appréciable des pertes d'impulsion par ionisation dans un absorbant donné. C'est ainsi que l'emploi d'un parallélépipède absorbant en polyéthylène (densité 0.92) de 46 cm de longueur a pour effet de réduire l'impulsion des protons de 160 MeV/c et celle des π^+ de 100 MeV/c, ce qui permet au second aimant de les séparer.

Si le décélérateur avait la forme usuelle d'un parallélépipède, le faisceau conserverait pratiquement la même dispersion en impulsion, Δp, à l'entrée et à la sortie de l'absorbant pour l'une et l'autre composantes. Admettons que l'impulsion du faisceau, supposé parallèle, varie linéairement d'une extrémité à l'autre de sa largeur l, en passant de P_0 a $P_0 + \Delta p$; on peut supprimer cet étalement en disposant sur le trajet du faisceau un absorbant complémentaire en forme de prisme triangulaire, présentant aux particules qui le tra-

versent une épaisseur d'autant plus grande que l'impulsion de celles-ci est plus élevée.

La monochromatisation ainsi obtenue est d'autant plus complète que la confusion en impulsion du faisceau traversant le prisme est plus réduite.
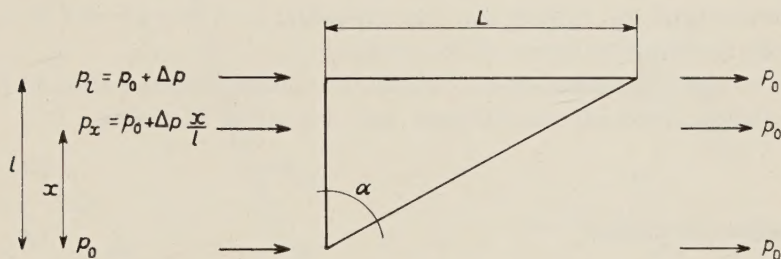

Fig. 2. – Prisme triangulaire destiné à supprimer l'étalement en impulsion d'un faisceau parallèle dont l'impulsion varie linéairement d'une extrémité à l'autre de sa largeur.

Soit k la perte d'impulsion par cm (qui dépend de la vitesse des particules et de la nature de l'absorbant), avec les notations évidentes de la Fig. 2, on doit avoir:

$$\Delta p = kL ,$$

d'où

$$L = \frac{\Delta p}{k} \qquad \text{et} \qquad \text{tg}\,\alpha = \frac{L}{l} = \frac{\Delta p}{kl} .$$

Dans le cas du faisceau de π^+ considéré et pour un absorbant en poly-éthylène, on a $k = 2.2$ MeV·c^{-1}·cm^{-1} ce qui, pour $\Delta p = 70$ MeV/c et $l = 6$ cm, donne $L = 31.8$ cm et $\text{tg}\,\alpha = 5.3$.

L'adjonction d'un tel prisme au parallélépipède décrit plus haut permet donc de monochromatiser sensiblement le faisceau (qui possédait initialement une bande d'impulsion étendue) et de concilier ainsi une bonne monochromatisation avec une grande luminosité.

Cette luminosité est cependant limitée par le fait que certaines des particules du faisceau subissent des interactions nucléaires dans le matériau du prisme. D'autre part, on utilisera, de préférence, un matériau de faible numéro atomique afin d'atténuer l'effet perturbateur de la diffusion multiple dans le décélérateur.

Enfin, des tubes en mylar remplis d'hélium sont disposés sur le trajet du faisceau pour diminuer la diffusion multiple.

2'2. *Entrée du faisceau dans la chambre.* – Nous avons étudié un échantillonnage de 1000 traces entrant dans la chambre.

738 d'entre elles entrent par la coupole. On considère les projections verticale (α) et horizontale (δ) des angles formés par une trace avec le grand axe de la chambre. α et δ ont des répartitions approximativement gaussiennes superposées à un faible fond continu.

En ne conservant que les traces pour lesquelles α et δ s'écartent de moins de 3 écarts-types des valeurs les plus probables ($-7°8 < \alpha < 0°4$ et $-3°8 < \delta < 5°2$) il reste 670 traces (67.0 ± 1.5)%.

Dans ce qui suit, seront seules considérées comme faisant partie du faisceau les trajectoires incidentes satisfaisant aux conditions ci-dessus.

3. – Analyse des clichés.

3'1. *Choix des événements à mesurer.* – L'examen des 70 321 paires de clichés stéréoscopiques obtenus a été effectué deux fois par des équipes indépendantes.

Les événements de création $\pi^+ + p \rightarrow \Sigma^+ + K^+$ recherchés doivent présenter une configuration en $\vee$ avec une cassure sur l'une des branches, celle du Σ^+ qui a une vie moyenne très courte, voisine de 10^{-10} s, et, éventuellement, une deuxième cassure ou une étoile sur l'autre branche celle, du K^+ s'il s'est désintégré dans la chambre (Fig. 3).
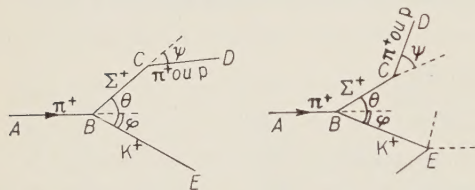

Fig. 3. – Configurations des événements de création $\pi^+ + p \rightarrow \Sigma^+ + K^+$.

Seuls sont retenus en vue de la mesure, ceux de ces événements qui ne présentent visiblement pas d'incompatibilité avec la cinématique $\pi^+ + p \rightarrow \Sigma^+ + K^+$.

La Fig. 4 reproduit des courbes de cinématique, relatives à la création associée $\pi^+ + p \rightarrow \Sigma^+ + K^+$, à la diffusion élastique $\pi^+ + p \rightarrow \pi^+ + p$ et à la diffusion élastique $p + p \rightarrow p + p$.

Un cas de création associée $\pi^+ + p \rightarrow \Sigma^+(n\pi^+) + K^+_{\pi_3}$, où le K^+ est un $\tau^+ \rightarrow \pi^+ + \pi^+ + \pi^-$, est reproduit par la Photo 1.

3'2. *Méthode de mesure.* – Les mesures ont été faites par la méthode graphique [8]. La courbure des trajectoires a été mesurée à l'aide d'arcs de cercles étalons.

[8] V. BORELLI, P. FRANZINI, I. MANNELLI, A. MINGUZZI-RANZI, R. SANTANGELO, F. SAPORETTI, V. SILVESTRINI, P. WALOSCHEK et V. ZOBOLI: *Nuovo Cimento*, **10**, 525 (1958).

Un programme pour l'ordinateur Mercury-Ferranti de Saclay a été établi; il permettait, à partir des mesures, de reconstituer l'événement (angles, longueurs et impulsions).
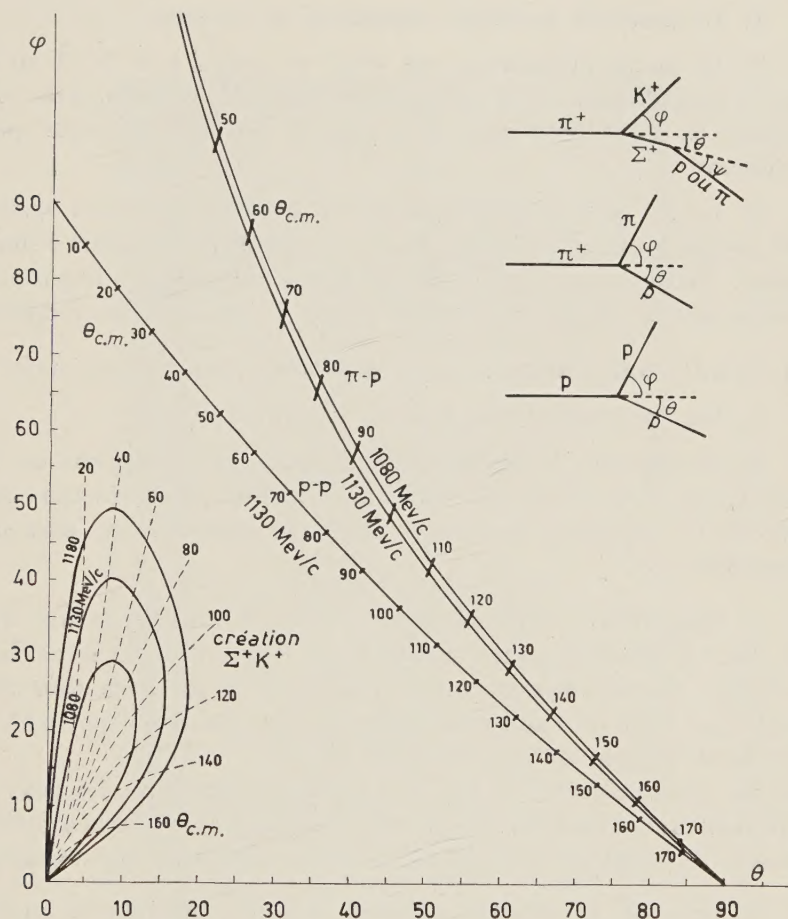
Fig. 4. — Courbes de cinématique relatives à la production associée $\pi^+ + p \to \Sigma^+ + K^+$, à la diffusion élastique $\pi^+ + p \to \pi^+ + p$ et à la diffusion élastique $p + p \to p + p$.

Les imprécisions ont été déterminées en tenant compte des erreurs expérimentales de mesure et de l'erreur introduite sur les courbures par la diffusion multiple.

Lorsque le K^+ s'arrête dans la chambre, la mesure de son parcours donne une valeur p_K plus précise, qui est alors retenue. De même, si la particule de désintégration du Σ^+ s'arrête dans la chambre, elle est immédiatement identifiée comme $\pi^+ (\to \mu^+ \to e^+)$, ou comme proton et la mesure de son parcours permet de remonter à l'impulsion p_Σ du Σ^+.

3˙3. *Critères de sélection des événements.* – Un événement n'est retenu que s'il satisfait aux conditions suivantes:

a) Conditions générales:

1) La particule incidente appartient au faisceau.

2) Le point d'interaction est situé au moins à 4 cm et au plus à 31 cm de la face d'entrée. Ceci est nécessaire pour avoir des mesures correctes d'orientation et d'impulsion aussi bien pour la particule incidente que pour les secondaires.

3) Le Σ^+ créé a vécu au moins $0.24 \cdot 10^{-10}$ s dans son système de référence propre (pendant ce temps, un Σ^+ de 520 MeV/c parcourt 3 mm dans le propane). Cette coupure est nécessaire pour éviter un biais important dans la détection des Σ^+ courts et une trop grande imprécision sur l'angle θ.

b) Conditions cinématiques à la création sur proton libre:

1) Les trois trajectoires doivent être coplanaires.

2) En principe, p_π incident étant donné, une seulement des quatre grandeurs p_K, p_Σ, θ, φ, suffit à déterminer la cinématique de la réaction. Lorsque plusieurs de ces quantités sont mesurables indépendamment, elles doivent être compatibles.

c) Conditions cinématiques à la désintégration du Σ^+. Pour un angle ψ donné et pour l'impulsion p_K, déterminée par la cinématique de la création, l'impulsion mesurée de la particule de désintégration doit être en accord avec l'une ou l'autre des valeurs calculables correspondant, soit à l'émission d'un proton, soit à celle d'un π^+.

Dans un certain nombre de cas, il y a compatibilité avec les deux modes de désintégration; il a toujours été possible d'en éliminer un par la considération de la densité de bulles sur la trajectoire de la particule de désintégration.

d) Densité de bulles: Elle ne doit présenter aucune incompatibilité avec la nature de la particule et l'impulsion qui lui est attribuée.

3˙4. *Contamination par des créations sur protons liés.* – Certaines des réactions $\pi^+ + p \to \Sigma^+ + K^+$ sur proton lié dans un noyau de carbone peuvent satisfaire aux critères et simuler un événement sur proton libre. Compte tenu de la largeur en impulsion du faisceau de π^+ et de la prévision des mesures, nous avons pu estimer par un calcul cette contamination à $(7 \pm 3)\%$. Cette valeur est compatible avec celle que l'on peut déduire des estimations de HANNUM *et al.* [5].

La contamination étant faible, on a admis qu'elle ne modifie pas les diverses répartitions, main on en a tenu compte dans le calcul de la section efficace.

A. Berthelot, A. Daudin, O. Goussu, F. Grard, M.-A. Jabiol, F. Lévy,
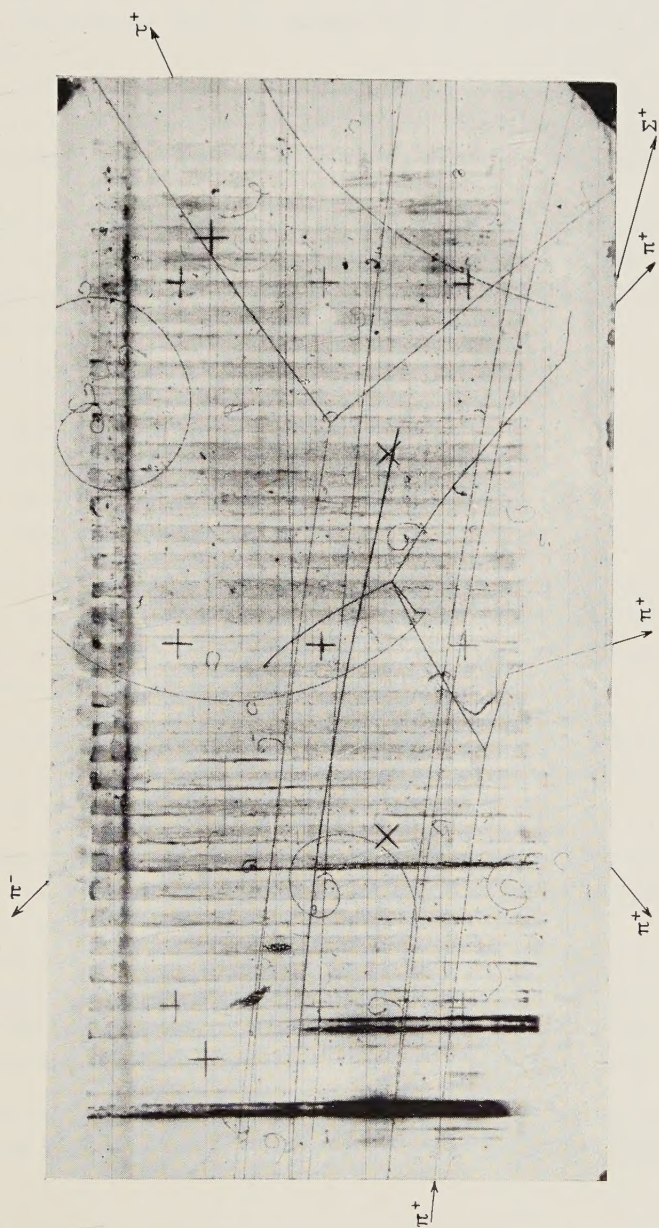C. Lewin, A. Rogozinski, J. Laberrigue-Frolow, C. Ouannès et L. Vigneron

Photo 1. – Création associée $\pi^+ + p \rightarrow \Sigma^+ (n\pi^+) + \tau^+ (\tau^+ \tau^+ \tau^-)$.

4. – Résultats expérimentaux et discussion.

Après application de tous les critères de compatibilité décrits, nous avons retenu 78 événements, soit 51 événements $\Sigma^+ \to \pi^+ + n$ et 27 événements $\Sigma^+ \to \pi^0 + p$.

Comme il est maintenant admis que le rapport de ces deux modes est égal à l'unité [9], le rapport observé 27/51, difficilement explicable par une fluctuation statistique, suggère une perte à la détection pour les événements $\Sigma^+ \to \pi^0 + p$.

4ˊ1. *Discussion des pertes.* – Une direction parallèle aux axes optiques des caméras de prise de vue est appelée « verticale »; on désigne par $\hat{N}$ l'angle qu'elle forme avec le « plan de production » AB, BC, BE (Fig. 3).

Pour déceler un événement, on doit voir:

1) une fourche en B,

2) un changement en C caractérisé
    — soit par une cassure $\widehat{BC, CD} = \psi$,
    — soit par un changement d'ionisation BC, CD,
    — soit par les deux à la fois.

a) Cas où le Σ^+ se désintègre en $\pi^0 + p$. Dans ce cas, l'angle ψ est toujours petit ($\psi_{max} = 25°$) et le changement d'ionisation est, en général, faible. Il en résulte une perte systématique de détection qui peut se trouver amplifiée dans différentes circonstances:

α) Lorsque $\hat{N}$ et $\theta + \varphi$ sont petits. La Fig. 5 qui représente la répartition des 27 événements observés en fonction de $\hat{N}$ pour deux bandes de valeur de $\theta + \varphi$, met cette perte en évidence.

β) Lorsque le parcours du Σ^+ est petit. Ceci aura en particulier pour conséquence de fournir une valeur trop élevée de la vie moyenne si on la détermine à partir des événements $\Sigma^+ (\pi^0, p)$.

Un second examen des clichés n'a pas permis d'augmenter de façon appréciable le nombre d'événements $\Sigma^+ \to (\pi^0, p)$ détectés. Nous en concluons que les configurations particulières mentionnées ci-dessus sont affectées d'une probabilité intrinsèque de détection très faible qui introduit un biais systématique.

b) Cas où le Σ^+ se désintègre en $\pi^+ + n$. Dans ce cas, d'une part, l'angle ψ peut atteindre 180°, d'autre part, le changement d'ionisation est, en

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[9] D. A. GLASER: *9th Annual Intern. Conference on High Energy Physics* (Kiev, 1959). p. 242.

général, important; des évènements de ce type échappent difficilement à l'observation.

Fig. 5. – Répartition des évènements observés en fonction de l'angle $\widehat{N}$ du plan de la création Σ^+K^+ avec la direction des axes optiques: a) répartition pour l'ensemble des évènements; b) répartition pour $\theta+\varphi \geqslant 40°$; c) répartition pour $\theta+\varphi < 40°$.

La Fig. 6 représente la statistique des 51 événements observés; les résultats sont statistiquement compatibles avec une perte nulle, sauf peut-être, pour $\widehat{N} < 10°$.

Pour $\widehat{N} > 10°$, il n'y a donc aucune raison de mettre en doute la valeur de l'efficacité de détection qui, après un double examen des clichés, a été trouvée pratiquement égale à un.

En conséquence, afin d'éliminer tous les cas où la détection est affectée de pertes systématiques, nous avons basé nos interprétations ultérieures uniquement sur les 49 cas où le Σ^+ se désintègre en π^+ et pour lesquels $\widehat{N}$ est supérieur à dix degrés.
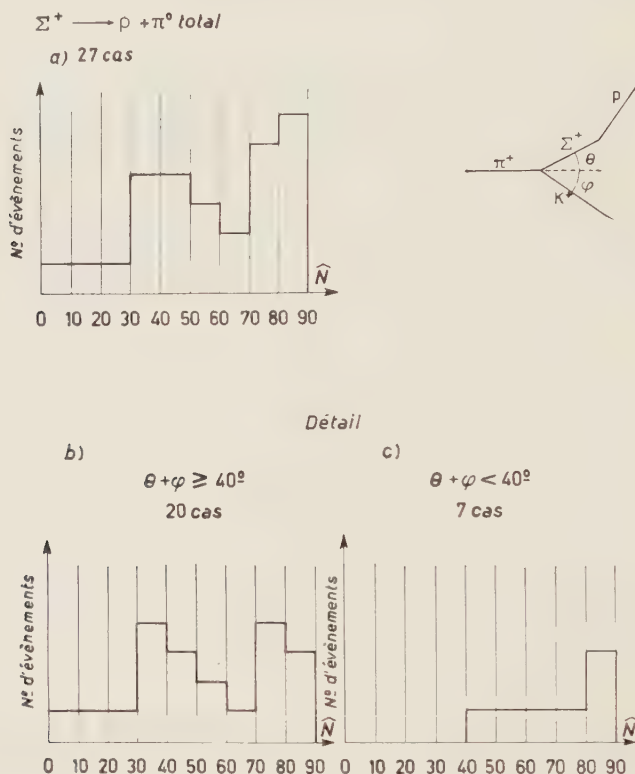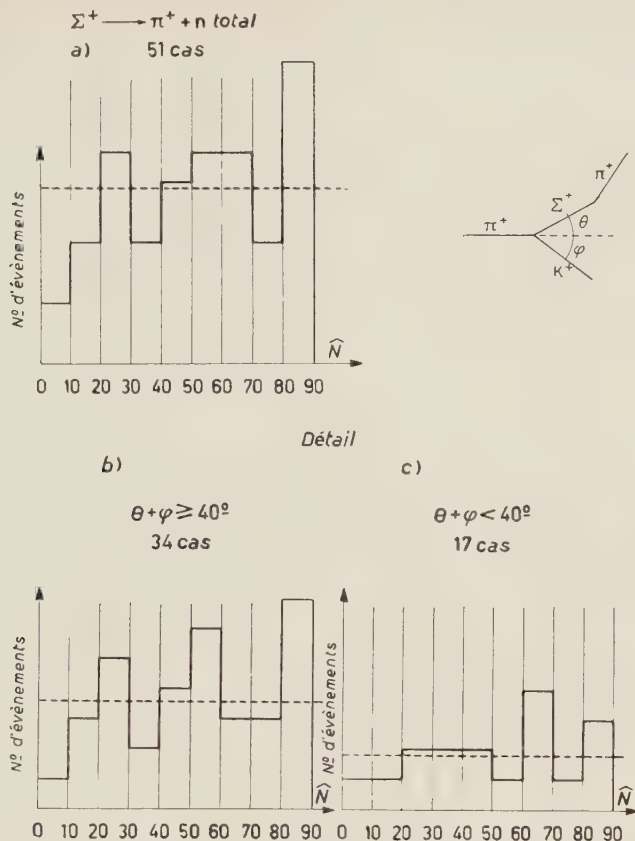
Fig. 6. – Répartition des évènements observés en fonction de l'angle $\widehat{N}$ du plan de création $\Sigma^+\mathrm{K}^+$ avec la direction des axes optiques: *a*) répartition pour l'ensemble des évènements; *b*) répartition pour $\theta + \varphi \geqslant 40°$; *c*) répartition pour $\theta + \varphi < 40°$.

4'2. *Calcul de la section efficace totale.*

a) **Calcul de la longueur effective de traces de π^+ étudiées:** Dans les clichés examinés au cours de notre étude, $N = 1\,042\,000$ traces entraient par la face d'entrée de la chambre. N a été estimé d'après un échantillonnage partiel auquel correspond une incertitude:

$$\varepsilon = \frac{\Delta N}{N} = 5 \cdot 10^{-3} \,.$$

1) La proportion de ces traces qui pénètrent par la coupole et ont une bonne orientation est:

$$k_1 = 0.67 \pm 0.015$$

avec une incertitude relative:

$$\varepsilon_1 = \frac{\Delta k_1}{k_1} = \frac{0.015}{0.67} = 2 \cdot 10^{-2} \ .$$

2) La proportion de π^+ dans le faisceau est:

$$k_2 = \frac{\pi^+}{\pi^+ + \mathrm{p} + \mu^+ + \mathrm{e}^+} = \frac{100}{100 + 10 + 8 + 1} = \frac{100}{119} = 0.84 \ ,$$

avec une incertitude relative

$$\varepsilon_2 = 2 \cdot 10^{-2} \ .$$

3) Certaines des traces de π^+ s'arrêtent dans la chambre par suite d'interactions. En prenant pour les π^+: $\sigma_{\mathrm{H}} = 29$ mb et $\sigma_{\mathrm{C}} = 240$ mb, et tenant compte de la densité (0.42) du propane détendu à une température de 62°, nous avons calculé le libre parcours moyen de π^+, soit

$$\lambda = 184 \text{ cm} \pm 20 \text{ cm} \ .$$

Compte tenu de la position du volume utile dans la chambre, définie plus haut, on voit alors que tout se passe comme si le nombre de particules était multiplié par

$$k_3 = 0.91$$

avec une incertitude relative

$$\varepsilon_3 = 1 \cdot 10^{-2} \ .$$

Le volume utile ayant une longueur $l = 27$ cm, la longueur effective des traces étudiées est:

$$L_\pi = N \cdot k_1 \cdot k_2 \cdot k_3 \cdot l = 14.4 \cdot 10^6 \text{ cm} \ .$$

Les incertitudes partielles étant indépendantes, l'incertitude sur L_π est:

$$\frac{\Delta L_\pi}{L_\pi} = \sqrt{\varepsilon^2 + \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2} = 3 \cdot 10^{-2} \ .$$

b) Calcul du nombre corrigé d'événements: A partir des 49 événements $\Sigma^+ \to \pi^+ + \mathrm{n}$ de vie propre $> 0.24 \cdot 10^{-10}$ s pour lesquels $10° < \hat{N} < 90°$, nous avons obtenu le nombre total d'événements $\pi^+ + \mathrm{p} \to \Sigma^+ + \mathrm{K}^+$ en multipliant par les facteurs correctifs suivants:

1) h_1 compensant l'abandon des événements $\Sigma^+ \to \pi^+ + n$ situés dans les plans $0° < \hat{N} < 10°$

$$h_1 = \frac{90}{80} = 1.125 \ .$$

2) h_2 compensant la perte des événements de vie propre $\tau < 0.24 \cdot 10^{-10}$ s $(= \tau_0)$

$$h_2 = \exp\left[\frac{\tau_0}{T}\right] = \exp\left[\frac{0.24}{0.75}\right] = 1.377 \ ,$$

l'incertitude relative étant:

$$\omega_2 = \frac{\Delta h_2}{h_2} = 3.5 \cdot 10^{-2} \ .$$

Pour calculer cette correction, nous avons pris pour vie moyenne d'un Σ^+ la valeur $T = (0.75^{+0.13}_{-0.09}) \cdot 10^{-10}$ s obtenue dans notre étude (le détail de cette détermination est donné plus loin).

3) h_3, pour tenir compte des deux modes de désintégration $\Sigma^+ \to \pi^+ + n$ et $\Sigma^+ \to \pi^0 + p$ qui sont également probables ([9]):

$$h_3 = 2.00 \ .$$

4) h_4, pour éliminer la contamination $(7 \pm 3)\%$ d'événements sur protons liés appartenant aux noyaux de carbone:

$$h_4 = 0.93$$

connu avec une incertitude relative

$$\omega_4 = 3 \cdot 10^{-2} \ .$$

Le nombre « vrai » d'événements de création $\pi^+ + p \to \Sigma^+ + K^+$ est donc:

$$n = 49 \cdot h_1 \cdot h_2 \cdot h_3 \cdot h_4 = 141.5 \ .$$

Les incertitudes ω_2 et ω_4 et l'incertitude relative $(14.5 \cdot 10^{-2})$ sur le nombre 49 sont indépendantes et s'ajoutent quadratiquement pour donner l'incertitude sur n:

$$\omega_n = \frac{\Delta n}{n} = 15 \cdot 10^{-2} \ .$$

Calcul de la section efficace totale:

$$\sigma_+ = \frac{n}{L_\pi} \cdot \frac{\text{masse moléc. de } C_3H_8}{8 \text{ (protons libres)} \times \text{densité } C_3H_8 \times 6.02 \cdot 10^{23}} \text{ cm}^2 ,$$

$$\sigma_+ = (0.214 \pm 0.035) \text{ mb} .$$

Cette valeur est bien compatible avec l'hypothèse de l'indépendance de charge. En effet, si l'on prend [2]:

$\sigma_0 = 0.39 \pm 0.037$ mb   pour la section efficace de la réaction (0) et

$\sigma_- = 0.27 \pm 0.028$ mb   pour la section efficace de la réaction (1). l'inégalité (3) prend la forme numérique:

$$0.88 \pm 0.04 \leqslant (0.52 \pm 0.03) + (0.46 \pm 0.04) .$$

4·3. *Section efficace différentielle*. – Soit $\Theta_{c.m.}$ l'angle d'émission du Σ^- dans le système du centre de masses. Suivant les valeurs de $\cos\Theta_{c.m.}$ les 49 événe-

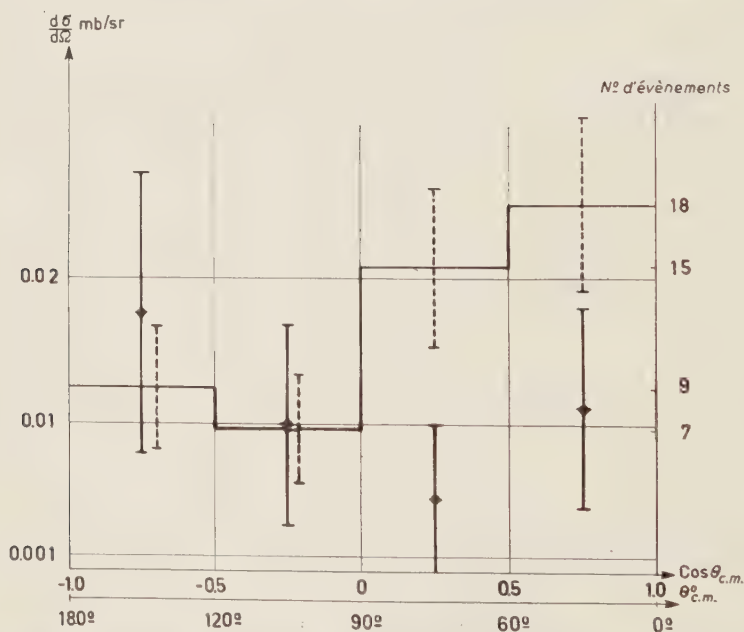Fig. 7.   Section efficace différentielle absolue basée sur les 49 événements $\Sigma^+ \to \pi^- + n$ ($\widehat{N} > 10°$). ├────┤ Nos valeurs. ─◆─ Limite inférieure calculée par CRAWFORD *et al.* [2] dans l'hypothèse de l'indépendance de charge.

ments retenus ont été classés en quatre groupes correspondant à des angles solides égaux à π stéradians. La Fig. 7 représente la section efficace différen-

tielle obtenue. Les barres d'erreurs correspondent à une incertitude $\sqrt{\bar{n}}$ sur n événements observés.

Nous avons également porté sur la même figure les limites inférieures de la section efficace différentielle avec leurs incertitudes, calculées par CRAWFORD et al. (²) à partir des sections efficaces différentielles des réactions (0) et (1) dans l'hypothèse de l'indépendance de charge. On constate que nos valeurs sont compatibles avec cette hypothèse.

L'échantillonnage est statistiquement insuffisant pour analyser, de façon précise, la distribution angulaire; cependant l'émission vers l'avant semble favorisée, le rapport

$$\frac{\text{avant}}{\text{arrière}} = \frac{33}{16},$$

différant significativement de l'unité.

4'4. *Vie moyenne des* Σ^+. – L'étude de la vie moyenne par la méthode de vraisemblance maximum pour les 49 cas $\Sigma^+ \rightarrow \pi^+ + n$ retenus donne la valeur:

$$\tau = \left(0.75^{+0.13}_{-0.09}\right) \cdot 10^{-10} \text{ s} .$$

Nous avons vu que ce type d'événements ne paraît pas affecté de pertes systématiques.

Si l'on considère maintenant les 27 événements $\Sigma^+ \rightarrow p + \pi_0$ ayant vécu plus de $0.24 \cdot 10^{-10}$ s on constate que la décroissance des Σ^+ se représente, mais assez médiocrement, par une exponentielle dont la vie moyenne serait: $\tau' = \left(0.95^{+0.23}_{-0.15}\right) \cdot 10^{-10}$ s. L'écart entre τ et τ' est dans le même sens et a le même ordre de grandeur que celui observé par FREDEN et al. (¹⁰). Il ne nous paraît pas possible de dire s'il correspond à un effet physique réel.

Nous avons vu, en effet, qu'on perd systématiquement certains événements $\Sigma^+ \rightarrow p - \pi_0$. Les chances de pertes augmentent quand la trace du Σ^+ est courte. Cela désavantage les Σ^+ ayant vécu peu de temps et doit produire un allongement apparent de la vie moyenne. Si l'on ne tient compte que des événements $\Sigma^+ \rightarrow p + \pi_0$, où le Σ^+ a vécu plus de $0.5 \cdot 10^{-10}$ s, les pertes diminuent, et l'on trouve $\tau'' = (0.76^{+0.22}_{-0.15}) \cdot 10^{-10}$ s, valeur compatible avec la vie moyenne τ déduite des événements $\Sigma^+ \rightarrow \pi^+ + n$.

La valeur $\tau = \left(0.75^{+0.13}_{-0.09}\right) \cdot 10^{-10}$ s qu'il nous paraît raisonnable d'admettre pour la vie moyenne du Σ^+ est comparée dans le Tableau I aux valeurs données par d'autres auteurs.

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(¹⁰) S. C. FREDEN, H. N. KORNBLUM et R. S. WHITE: *Nuovo Cimento*, **16**, 611 (1960).

TABLE I. – *Valeurs de la vie moyenne τ des Σ^+.*

Auteurs	Nombre d'événements	$(\tau \cdot 10^{10})$ s	Auteurs	Nombre d'événements	$(\tau \cdot 10^{10})$ s
Emulsions $\Sigma^+ \to p + \pi^0$ (vol)			$\Sigma^+ \to p + \pi^0$ (vol+repos)		
KIEV	99	$0.75^{+0.42}_{-0.20}$	GIBS	32+27	$0.94^{+0.17}_{-0.12}$
KAPLON	25	$0.49^{+0.31}_{-0.17}$	KIEV	80+99	$0.99^{+0.11}_{-0.09}$
	59	$0.50^{+0.17}_{-0.11}$	KAPLON partiels	25+29	$0.80^{+0.20}_{-0.13}$
FREDEN	49	$0.95^{+1.87}_{-0.38}$		59+72	$0.79^{+0.11}_{-0.11}$
K⁻–Coll.	31	$0.68^{+0.41}_{-0.19}$	FREDEN	49+68	$0.85^{+0.14}_{-0.11}$
Chambres à bulles $\Sigma^+ \to \pi^+ + n$ et $\Sigma^+ \to p + \pi^0$			K⁻-Coll.	19+22	$0.82^{+0.34}_{-0.20}$
ALVAREZ	58	0.70 ± 0.10			
GLASER	19	$0.95^{+0.37}_{-0.23}$			
Notre travail	49 ($\pi^+ + n$)	$0.75^{+0.13}_{-0.09}$			

Les valeurs de ce tableau sont tirées du rapport de Glaser à la IX° Conférence des Hautes Energies (Kiev, 1959), et des articles de KAPLON *et al.* ([11]) et de FREDEN *et al.* ([10])

4'5. *Asymétries dans la désintégration du Σ^+.* – L'étude de la distribution angulaire des particules de désintégration du Σ^+ peut donner différentes informations.

Dans le système du centre de masses, avec les définitions habituelles, les asymétries haut-bas peuvent donner des informations sur la non-conservation de la parité dans les interactions faibles (désintégration de l'hyperon). Les asymétries avant-arrière et droite-gauche peuvent donner des informations sur la conservation de la parité dans les interactions fortes (création de l'hyperon).

Nous obtenons pour les $\Sigma^+ \to \pi^+ + n$

$$\frac{\text{haut}}{\text{bas}} = \frac{22}{27} \to 2\,\frac{\text{haut} - \text{bas}}{\text{haut} + \text{bas}} = 0.20 \pm 0.28 \,,$$

$$\frac{\text{avant}}{\text{arrière}} = \frac{26}{23} \to 2\,\frac{\text{avant} - \text{arrière}}{\text{avant} + \text{arrière}} = 0.12 \pm 0.28 \,,$$

$$\frac{\text{droite}}{\text{gauche}} = \frac{29}{20} \to 2\,\frac{\text{droite} - \text{gauche}}{\text{droite} + \text{gauche}} = 0.37 \pm 0.28 \,.$$

([11]) M. F. KAPLON, A. C. MELISSINOS et T. YAMANOUCHI: *Ann. Phys. U.S.A.*, **9**, 139 (1960).

Les incertitudes indiquées correspondent à un écart type. Il n'apparaît pas d'asymétrie significative; la valeur obtenue pour l'asymétrie haut-bas est compatible avec la valeur de CRONIN et al. [12].

<p style="text-align:center">* * *</p>

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[12] J. W. CRONIN, B. CORK, L. KERTH, W. A. WENZEL et R. L. COOL: *Bull. Am. Phys. Soc.*, **5**, 11 (1960); R. L. COOL, B. CORK, J. W. CRONIN et W. A. WENZEL: *Phys. Rev.*, **114**, 912 (1959).

<p style="text-align:center">RIASSUNTO (*)</p>

Abbiamo analizzato 78 eventi di produzione associata $\pi^+ + p \to \Sigma^+ + K^+$, per mezzo di una camera a bolle a propano (40 cm $\times$ 20 cm $\times$ 20 cm) posta in un campo magnetico di 12.7 kG. Il fascio usato aveva un impulso di (1.155 ± 0.025) GeV/c ed aveva la seguente composizione: $\pi^+ : p : \mu^+ : e^+ = 100 : 10 : 8 : 1$. Abbiamo ottenuto i seguenti risultati: 1) le sezioni d'urto totale $(\sigma = (0.214 \pm 0.035)$ mb) e differenziale ottenute si accordano con l'ipotesi della indipendenza della carica; 2) per gli eventi $\Sigma^+(\pi^+ n)$ non ho trovato significative asimmetrie sopra-sotto, destra-sinistra od avanti-indietro; 3) la vita media del Σ^+ è $\tau = (0.75^{+0.13}_{-0.09}) \cdot 10^{-10}$ s.

(*) *Traduzione a cura della Redazione.*

Field Metrics.

III. Field Charges.

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Summary. — The determinants of the field metrics which have been defined earlier by means of a correlation to the field wave equations are evaluated and found to be products of the strengths of the field charges. It is now suggested that these determinants have the value unity in the natural field units and the resulting correlations with principles of the theory of relativity are mentioned. It is shown that then generally the strengths of all field charges can be determined. The procedure is illustrated in detail for the Tanikawa field theory and the obtained values of the field charges are found to support it.

1. – From the physical significance of a metric for a quantum field and from the similarity between the expressions for the commutation relations, Feynman propagator and the metric of the free photon field we suggested [1] a hypothesis that there exist metric equations for all quantum fields and postulated that,

1) In accordance with the principles of relativity a quantum field has a metric which is correlated to its commutation relations and Feynman propagator. Conversely by means of this correlation the metric determines the field wave equation which is found [1] to be non local due to the occurrence of internal degrees of freedom in it, but otherwise the metric is arbitrary and can be so chosen as to obtain a given wave equation.

We should now like to utilize another principle of relativity, the gauge invariance [2] and show that the strengths of the field charges can be derived

[1] P. Sen: *Nuovo Cimento*, **15**, 513 (1960); **18**, 613 (1960).
[2] H. Weyl: *Space-Time-Matter* (New York, 1950).

by it from a given set of field metrics. Furthermore by comparing the values
of these calculated charges with the known charges, the electron charge e,
the strong spin 0 meson charge e_s, the intermediate weak meson ([3]) charge e_w
and the Fermi β interaction constant G a unique set of metrics or wave equa-
tions can be obtained and some of the present ambiguities in the wave equa-
tions such as their order, a pseudovector or pseudoscalar e_s, vector or scalar
e_w, and e_w or G are eliminated.

It has often been suggested ([4]) that the fields and their charges could be
obtained from a generalization of the quantum field gauge invariance. In
continuation of our basic idea of correlating classical and quantum fields by
means of the fundamental principles of relativity we shall utilize a classical
form of the principle of gauge invariance here and accordingly make the se-
cond postulate,

2) The volume densities of the quantum fields obtained from the deter-
minants of their metrics are equal to unity in the natural field units ([5]) in which
the electromagnetic interaction constant is $4\pi hce^{-2}\lambda_e^2 = 1$. This postulate is a
principle of the conformal projective theories of general relativity ([6]) and is
equivalent to the original form of the principle of gauge invariance ([7]).

After deducing the metrics of a given set of wave equations when their
determinants are evaluated they are found to be simple and generally a dif-
ferent product of the field charges for every metric. Thus the second postulate
provides a remarkably simple procedure for the determination of the field
charges and since the number of metrics exceeds the number of charges some
verification of the calculations is also obtained. However if the product of
two charges occurs in the same way in all the determinants it is necessary
to use the known value of one of them to obtain the calculated value of the
other. We shall also show later in the Section 4 how the second postulate
may be reformulated so that all measurement units are included.

([3]) Y. TANIKAWA and S. WATANABE: *Phys. Rev.*, **113**, 1344 (1959); S. ONEDA and
Y. TANIKAWA: *Phys. Rev.*, **113**, 1354 (1959). See also T. KINOSHITA: *Phys. Rev. Lett*,.
4, 378 (1960); P. T. MATTHEWS and A. SALAM: *Nuovo Cimento*, **19**, 837 (1961).

([4]) A. SALAM and J. C. WARD: *Nuovo Cimento*, **11**, 568 (1959); **19**, 165 (1961).

([5]) The transformations $A_\mu \to \lambda^{-1}A_\mu$, $\psi \to \lambda^{-1}\psi$, in the Dirac-Maxwell equations ([1])
transforms them to the natural field units. The requirement of the metric formulation
that the interaction constants appear in the same form in all the wave equations
determines uniquely the transformations for all the other fields. When these trans-
formations have been made the wave and metric equations in the natural field units
are obtained. We note that to obtain the natural units here $\lambda \to 1$, $e_s \to \lambda^{-1}e_s$, $e_w \to \lambda^{-1}e_w$.

([6]) T. Y. THOMAS: *The Differential Invariants of Generalized Spaces* (Cambridge,.
1934), pp. 20, 21, 66.

([7]) P. G. BERGMANN: *Introduction to the Theory of Relativity* (New York, 1947), p. 246,.

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Now in the next Section 2 and the Section 3 the procedure for calculation of the charges is illustrated in detail for the Tanikawa field theory [3] in which there exist the charges e, the pseudoscalar meson charge e_s and the chiral scalar neutral and charged meson charges e_w and $e_{w'}$: e_w and $e_{w'}$ are of the same magnitude so for simplicity and since when they are different our argument can readily be extended without affecting the general conclusions we let $e_w = e_{w'}$. We also note here that the charge strengths proposed for this field theory are supported by their calculated values and that it seems difficult to obtain such agreement for the other possible field theories.

2. – Let the lepton (baryon) spin $\frac{1}{2}$ fields ψ_ν and ψ_e (ψ_N and ψ_D) be combined so that

$$(1) \qquad \psi_L = \begin{pmatrix} \psi_\nu \\ \psi_e \end{pmatrix}, \qquad \psi_B = \begin{pmatrix} \psi_N \\ \psi_D \end{pmatrix}, \qquad \psi = \begin{pmatrix} \psi_L \\ \psi_B \end{pmatrix},$$

and let the metric equation for $\psi(x, \tau)$ be given by

$$(2) \qquad \overline{\psi}(x, \tau)\Big\{ x^2 + \tau^2 + \lambda_e^2\, \overline{A_\mu(x)\, A_\mu(x)} +$$
$$+ \sum_{q=+,0,-} \big(\lambda_s^2\, \overline{\pi^q(x,\tau)\pi^q(x,\tau)} + \lambda_w^2\, \overline{\chi^q(x,\tau)\,\chi^q(x,\tau)}\big)\Big\}\psi(x,\tau) = 0 .$$

The notation of reference [1] is used here, $\lambda_s^2 = e_s^2/4\pi\hbar c$, $\lambda_w^2 = e_w^2/4\pi\hbar c$; π^q and $\varkappa^q$ are pseudoscalar parity conserving and scalar chiral meson fields respectively.

The metric of the spin $\frac{1}{2}$ fields is given by the expression in curly brackets [1] in (2) and x_μ, τ_μ, $\widetilde{A}_\mu$, $\widetilde{\pi}^q$ and $\widetilde{\chi}^q$ are to be considered the independent variables of the metric so that it can be written in the form $g_{\varrho\sigma}(\psi)\overline{x}_\varrho x_\sigma$ where $x_\sigma \equiv$ $\equiv (x_\mu, \tau_\mu, \widetilde{A}_\mu, \widetilde{\pi}^q, \widetilde{\chi}^q)$ and $\varrho, \sigma = 1, ..., 18$. We note that $g_{\varrho\sigma}(\psi)$ is a diagonal matrix, $g_{\varrho\sigma}(\psi) \sim \delta_{\varrho\sigma}$, and obtain from the second postulate when $\lambda_e \to 1$, $\lambda_s \to \lambda_e^{-1}\lambda_s$, $\lambda_w \to \lambda_e^{-1}\lambda_w$,

$$(3) \qquad |g(\psi_{\varrho\sigma})|^{\frac{1}{2}} = \lambda_e^{-6}\lambda_s^3\lambda_w^3 = 1 , \qquad \lambda_e^{-2}\lambda_s\lambda_w = 1 ,$$

in natural field units. Inserting the known value [3] of $\lambda_s^2 = 15.0$ we obtain $\lambda_w^2 = 3.5\cdot10^{-6}$ and the Fermi β interaction constant whose known value is $1.0\cdot10^{-5}/m_N^2$, is found [3] to be $2.23\cdot10^{-5}/|m_\chi^2 - m_N^2|$ where m_χ and m_N are the chiral meson and nucleon masses. Therefore we obtain $m_\chi \approx 3300$ electron masses which is in accord with the Tanikawa field theory.

We should now like to show that the Tanikawa field fermion wave equa-

tions can be obtained from the metric equation (2) in which all the fermion fields ψ_ν, ψ_e, ψ_N and ψ_p have had to be combined in the form ψ on account of their interaction with each other through the π and χ meson fields. A second order wave equation is obtained for the field from the metric eq. (2) by means of the first postulate ([1]). We note that it does not contain the fermion intrinsic magnetic moment term and that the metric in (2) can easily be modified to include it. But at present we wish for the simplest formulation and have neglected all but the principal terms and expect that the results will not be appreciably altered by such omission ([8]). When the second order wave equation is reduced to first order wave equations the spin matrices γ_μ, the isobaric spin matrices σ_i, where $i = 1, ..., 3$, which decompose ψ_L (ψ_B) into ψ_ν and ψ_e (ψ_N and ψ_p), and the isotopic spin matrices τ_i which decompose ψ into ψ_L and ψ_B will occur. We expect that the fermion meson interaction term in the first order wave equations can be written in the form $\sigma_i \times \tau_i q_{ij}$, where q_{ij} is a symmetric tensor whose six components are π^q and χ^q and that the Tanikawa couplings will be obtained from this interaction term. When the second order wave equations are constructed from these first order wave equations we expect the occurrence of the intrinsic magnetic moment type cross term which however can be neglected for the present ([8]). The first order wave equations are not investigated further here as we are primarily interested in the metric eq. (2). However of course the eq. (2) has a meaning only if there exists the suggested form of interaction.

3. – In the Tanikawa field theory there exist two families of spin 0 mesons, the π-mesons of masses 270, 970, ..., and the mesons of masses 3 300, ..., electron masses ([9]) and we can combine them just like the fermions in eq. (1),

$$(4) \qquad \varphi^+ = \pi^+ \chi^+ , \qquad \varphi^0 = \pi^0 \chi^0 , \qquad \varphi^- = \pi^- \chi^- .$$

([8]) The most convenient way to introduce the term corresponding to the fermion intrinsic magnetic moment in their metric equation without altering the already found value of the determinant $|g_{\varrho\sigma}(\psi)|$ is to add the term $2\lambda\gamma_\mu\gamma_\nu x_\mu A_\nu$, where $\mu < \nu$, to the metric. Then the matrix becomes triangular and its determinant is determined by its diagonal terms only.

([9]) The mass spectrum of the mesons and fermions as well as the selection rules which allow their relatively long life times by forbidding the strong interactions can be obtained from the internal co-ordinate wave equations (P. SEN: *Nuovo Cimento*, **8**, 407 (1958)). It is interesting to note that now there will exist similar selection rules for the weak Tanikawa interactions also; that the known elementary particle decays must be allowed by them; that some types of decays will also be forbidden by them; that the decays allowed by means of chiral mesons of different masses will have different Fermi interaction constants; so that perhaps a synthesis of all weak decays will be possible.

Then let the spin 0 meson metric equations be [1]

$$(5) \qquad \varphi^0 \{x^2 + \tau^2 + \lambda_w^2 \, \mathrm{Sp} \, (\gamma_5 \widetilde{K}_{N'} \gamma_5 \widetilde{K}_\nu^c + \gamma^5 \widetilde{K}_\mathrm{p} \gamma \widetilde{K}_\mathrm{e}^c) +$$
$$ + \lambda_s^2 \, \mathrm{Sp} \, (\gamma_5 \widetilde{K}_{N'} \gamma_5 \widetilde{K}_{N'}^c + \gamma_5 \widetilde{K}_\mathrm{p} \gamma_5 \widetilde{K}_\mathrm{p}^c) \} \varphi^0 = 0 \, ,$$

$$(6) \qquad \overline{\varphi}^+ \{x^2 + \tau^2 + \lambda_e^2 \widetilde{A}_\mu \widetilde{A}_\mu + \lambda_w^2 \, \mathrm{Sp} \, (\gamma_5 \widetilde{K}_\mathrm{e} \gamma_5 \widetilde{K}_\nu^c) + \lambda_s^2 \, \mathrm{Sp} \, (\gamma_5 \widetilde{K}_\mathrm{p} \gamma_5 \widetilde{K}_{N'}^c) \} \varphi^+ = 0 \, ,$$

where the co-ordinates of the fields have not been explicitly mentioned and K's are the fermion propagators. These metric equations contain the Tanikawa field interactions explicitly and from the corresponding second order wave equations we can obtain in a straightforward way the second order wave equations for the π and χ mesons which correspond to the Tanikawa field theory. Then in that case, as in Section 2 we are only interested in their metrics which are again given by the expressions in curly brackets in the eq. (5) and (6). The independent variables in these metrics are x_μ, τ_μ, $\widetilde{A}_\mu$, $\widetilde{K}_{\mu\nu}$ and the metric can again be written in the form $g_{\varrho\sigma}(q^q) \widetilde{Y}_\varrho Y_\sigma$, where $Y_\varrho \equiv$ $\equiv (x_\mu, \tau_\mu, \widetilde{A}_\mu, \widetilde{K}_{\mathrm{e}, \mu\nu}, ..., \widetilde{K}_{\mathrm{p}, \mu\nu})$. The determinant $g_{\varrho\sigma}(q^q)|$ is easily evaluated specially by noting that $\widetilde{K}_{\mu\nu} \sim \delta_{\mu\nu}$ so that $g_{\varrho\sigma}(q^q)$ becomes a diagonal matrix. Then we again obtain $\lambda_e^{-2} \lambda_s \lambda_w = 1$ and note that this agreement with the eq. (3) is the result of the structural symmetry of the fermion and meson metric eq. (2), (5) and (6) obtained by means of the eq. (1) and (4). It has the consequence that the meson wave equations are of second order.

Finally let the photon metric equation be

$$(7) \qquad A_\mu \{x^2 + \lambda_e^2 (\widetilde{\pi}^+ \widetilde{\pi}^+ + \widetilde{\pi}^- \widetilde{\pi}^- + \widetilde{\chi}^+ \widetilde{\chi}^+ + \widetilde{\chi}^- \widetilde{\chi}^-) \} A_\mu +$$
$$ + A_\mu \{ \lambda_e^2 \, \mathrm{Sp} \, (\gamma_\mu \widetilde{K}_\mathrm{e} \gamma_\nu \widetilde{K}_\mathrm{e}^c + \gamma_\mu \widetilde{K}_\mathrm{p} \gamma_\nu \widetilde{K}_\mathrm{p}^c) \} A_\nu = 0$$

which has been written in a more convenient form and differently from before [1]. The metric for the photon field A_μ is obtained by noting that according to the first postulate $[A_\mu, A_\nu] \sim \delta_{\mu\nu}$ so that the index ν in the second term of (7) is replaced by μ. The matrix $g_{\varrho\sigma}(A_\mu)$ is seen to be diagonal $g_{\varrho\sigma}(A_\mu) \sim \delta_{\varrho\sigma}$ and its determinant is easily evaluated. The second postulate then gives the identity $1 = 1$ which verifies the consistency of the metric equations or of the wave equations.

4. – To transform the above results to other units than the natural field units it is necessary that the second postulate be so reformulated that the volume density of a given metric is equal to α^n, where α is a constant and n is equal to the total number of components of the interacting fields in the metric. For example, the eq. (3) is then replaced by $|g_{\varrho\sigma}(\psi)|^{\frac{1}{2}} = \lambda_e^4 \lambda_s^3 \lambda_w^3 = \alpha^{10}$

and we note that the field units are α times the natural units. Thus the use of units other than the natural units has the disadvantage of the restatement of the second postulate for every metric but it also indicates that further re-formulations for the second postulate are also possible and that it could be so stated that any field theory is compatible. However the above statement of the second postulate is the most natural and suggests the correlations mentioned in Section **1**.

* * *

RIASSUNTO (*)

Si valutano i determinanti delle metriche di campo precedentemente definite per mezzo di una correlazione con le equazioni d'onda del campo e si trova che sono i prodotti delle intensità delle cariche di campo. Si postula ora che tali determinanti abbiano valore unitario nelle unità naturali del campo e si rilevano le correlazioni risultanti coi principi della teoria della relatività. Si dimostra che, in generale, allora si possono determinare le intensità di tutte le cariche del campo. Si illustra in dettaglio il procedimento per la teoria del campo di Tanikawa e si trova che i valori delle cariche del campo trovati ne sono una prova.

(*) *Traduzione a cura della Redazione.*

On the Analysis of Deuteron Stripping Reactions.

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Summary. — The equivalence of the Butler approximation and the Born approximation for the deuteron reaction in the extrapolation method of determining the reduced width of the captured nucleon is verified, and the relation of the extrapolation method to the method of Bowcock is discussed. Numerical comparison is given for the particular case of $C^{12}(d, p) C^{13*}(3.09 \text{ MeV state})$ at $E_d = 2.89 \text{ MeV}$.

1. – It is usual to analyse the angular distribution of a given (d, p) or (d, n) reaction in terms of the Butler formula [1] of the deuteron stripping reaction. One gets the value of the orbital angular momentum of the captured nucleon by fitting the position of the maximum of the theoretical angular distribution given by the Butler formula to the position of the maximum of the experimental angular distribution with a reasonable value of the radius of the target nucleus which is an adjustable parameter in the Butler formula. The value of the reduced width of the captured nucleon is determined by normalizing the theoretical cross-section to the experimental one at the peak.

Since, in some cases, the theoretical angular distribution due to the Butler theory does not fit well the experimental one, and the value of the reduced width determined by the normalization method differs from the value determined from other reactions, there have been many efforts [2] to study the

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[1] See for example, S. T. BUTLER: *Nuclear Stripping Reactions* (New York, 1957).
[2] See for example, G. BREIT: *Handbuch der Physik* (Berlin, 1959), Sect. 45.

effects of the interactions such as the final state interaction and the Coulomb interaction which are not taken into account in the Butler approximation. Since the absolute value of the theoretical cross-section in those refined calculations depends on the interference between the Butler theoretical reaction amplitude and the reaction amplitudes due to the residual interactions, the value of reduced width determined by the normalization method is different from that deduced from the analysis in terms of the Butler formula and depends on the way of taking into account the residual interactions in a complicated way.

Recently AMADO [3] has applied to the analysis of deuteron stripping reactions the method of extrapolation proposed by CHEW and LOW [4] for the analysis of a type of reaction in which a single particle is exchanged between two incoming particles. He has noted that the reaction amplitude due to the Butler approximation has a pole as a function of the momentum transfer in the unphysical region. The method of extrapolation of the experimental angular distribution up to an angle with $\cos \theta > 1$, which corresponds to the imaginary momentum transfer at the pole, gives the value of the reduced width as the residue of the cross-section at the pole.

The purpose of this note is two-fold. One is to point out that the position and residue of the Butler reaction amplitude are same as the position and residue of the reaction amplitude due to the Born approximation, confirming the general idea of CHEW and LOW. Second is to discuss the relation of the extrapolation method to the method of analysis proposed by BOWCOCK [5].

2. – For definiteness, let us take the case of (d, p) reactions. The reaction amplitude is, apart from some kinematical constants, given by the product of two factors [6]: one is the probability amplitude for finding the final proton momentum k_p in the initial deuteron

$$(1) \qquad P(k_p - \tfrac{1}{2} k_d) = \int dr \, \psi_d(r) \exp \left[i(k_p - \tfrac{1}{2} k_d) \cdot r \right],$$

where k_d is the momentum of the incident deuteron and ψ_d is the wave function of the deuteron, r being the relative co-ordinate between the proton and neutron. With the Yukawa type wave function for ψ_d, (1) becomes

$$(2) \qquad P(k_p - \tfrac{1}{2} k_d) = 4\pi N / [(k_p - \tfrac{1}{2} k_d)^2 + \alpha^2],$$

[3] R. D. AMADO: *Phys. Rev. Lett.*, **2**, 399 (1959).

[4] G. F. CHEW and F. E. LOW: *Phys. Rev.*, **113**, 1640 (1959).

[5] J. E. BOWCOCK: *Proc. Phys. Soc.* (*London*), **68** A, 512 (1955).

[6] P. B. DAITCH and J. B. FRENCH: *Phys. Rev.*, **87**, 900 (1952).

where α^2 is related to the binding energy, B_d, of the deuteron through the relation $\alpha^2 = M B_d/\hbar^2$, M being the nucleon mass, and N is the normalization constant of the deuteron wave function. This factor has a pole as a function of the square of the momentum transfer of the proton $K = k_p - \frac{1}{2}k_d$ when $K^2 = -\alpha^2$. It is easy to see that the denominator of (2) can also be expressed as $q^2 + \varkappa_n^2$, where q is the momentum of the recoiled neutron to be captured, $q = k_n^2 - (M_i/M_f)k_p$, M_i and M_f being the masses of the initial and final nuclei respectively and $\varkappa_n^2$ is related to the separation energy S_n of the captured neutron from the final nucleus through the relation $S_n = \hbar^2\varkappa_n^2/2 M_n$, M_L being the reduced mass of the neutron $M_n = M(M_i/M_f)$. From the energy conservation we have

$$(3) \qquad \frac{\hbar^2 k_d^2}{2 M_d} - \frac{\hbar^2 k_p^2}{2 M_p} - \frac{\hbar^2 \alpha^2}{M} + \frac{\hbar^2 k_n^2}{2 M_n} = 0 \,,$$

with $M_d = 2 M M_i/(2 M - M_i)$ and $M_p = M M_f/(M - M_f)$. From this relation and the definitions of q and K, we get

$$(4) \qquad 2(K^2 + \alpha^2) = (M_f/M_i)(q^2 + \varkappa_n^2) \,.$$

Thus the zero of $(K^2 + \alpha^2)$ is same as that of $(q^2 + \varkappa_n^2)$ which is used to define the pole of the Butler reaction amplitude by AMADO.

The other factor is the probability amplitude for the capture of the neutron from the plane wave state $\exp[i q \cdot r_n]$ to the final bound state:

$$(5) \qquad I_{\text{Born}} = \int_0^\infty dr_n \int d\xi \chi_f^*(r_n, \xi) \, V_{n\xi}\chi_i(\xi) \exp[i q \cdot r_n] \,,$$

where χ_i and χ_f are the wave functions of the initial and final nuclei respectively, ξ representing collectively all the co-ordinates of the initial nucleus, and $V_{n\xi}$ is the interaction potential between the neutron and the initial nucleus. Making use of the Schrödinger equation satisfied by χ_f and $\chi_i \exp[i q \cdot r_n]$, (5) can be rewritten as

$$(6) \qquad I_{\text{Born}} = -\frac{\hbar^2}{2 M_n}(q^2 + \varkappa_n^2) \int_0^\infty dr_n \exp[i q \cdot r_n]f(r_n) \,,$$

with

$$(7) \qquad f(r_n) = \int d\xi \chi_f^*(r_n, \xi) \, \chi_i(\xi) \,.$$

In the Butler approximation, the integral with respect to r_n from zero to a,

the nuclear radius, is neglected supposing that the neutron wave function f oscillates rapidly for $r_n \leqslant a$ because of the interaction between the neutron and the nucleons in the initial nucleus so that it averages to zero. Thus we have

$$(8) \qquad I_{\text{Butler}} = -\frac{\hbar^2}{2M_n}(q^2 + \varkappa_n^2)\int_a^\infty d\mathbf{r}_n \exp(i\mathbf{q}\cdot\mathbf{r}_n)\, f(\mathbf{r}_n)\,.$$

Using again the Schrödinger equations satisfied by $f(\mathbf{r}_n)$ for $r_n > a$ and $\exp[i\mathbf{q}\cdot\mathbf{r}_n]$, and applying the theorem of Green, (8) becomes

$$(9) \qquad I_{\text{Butler}} = -\frac{\hbar^2 a^2}{2M_n}\int d\Omega_n\left[f\frac{\partial}{\partial r_n}\exp[i\mathbf{q}\cdot\mathbf{r}_n] - \exp[i\mathbf{q}\cdot\mathbf{r}_n]\frac{\partial}{\partial r_n}f\right]\Bigg|_{r_n=a}\,.$$

Now, we define the average potential $\langle V \rangle$ to be

$$(10) \qquad \langle V \rangle = \frac{\int_0^a d\mathbf{r}_n\int d\xi\, \chi_f^* V_{n\xi}\, \chi_i \exp[i\mathbf{q}\cdot\mathbf{r}_n]}{\int_0^a d\mathbf{r}_n \exp[i\mathbf{q}\cdot\mathbf{r}_n]f}\,.$$

With this $\langle V \rangle$ we can rewrite (6) using the Schrödinger equation as

$$(11) \qquad I_{\text{Born}} = \frac{-\langle V \rangle}{\langle V \rangle + \hbar^2(q^2 + \varkappa_n^2)/2M_n}\cdot\frac{\hbar^2}{2M_n}(q^2 + k_n^2)\int_a^\infty d\mathbf{r}_n \exp[i\mathbf{q}\cdot\mathbf{r}_n]f\,,$$

thus we have

$$(12) \qquad I_{\text{Born}} = \frac{\langle V \rangle}{\langle V \rangle + \hbar^2(q^2 + \varkappa_n^2)/2M_n} I_{\text{Butler}}\,.$$

It is clear that when q^2 tends to $-\varkappa_n^2$, I_{Born} tends to I_{Butler}. Thus at the pole the Born and Butler reaction amplitudes take same value of the residue.

3. – The basic physical idea in the Butler theory is that since the wave function of the deuteron has a rather large spacial extension there is a great probability that the proton in the incident deuteron passes outside the target nucleus while the neutron comes into the target nucleus to be captured, though in the actual derivation of the Butler formula the initial and final wave functions of the proton are taken to be plane waves of infinite extension. According to this basic idea, BOWCOCK (^{7}) has proposed a method of analysis

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(^{7}) For the modification of the Butler formula by the condition that in a simple stripping (d, p) reaction the proton must be outside the nucleus, and the comparison of the modified formula with experiments, see T. HONDA and M. NAGASAKI: *Proc. Phys. Soc. (London)*, **74** A, 517 (1959).

of deuteron stripping reactions. In Bowcock's method, the experimental and the Butler reaction amplitudes normalized to the experimental one at the peak are expanded in terms of the outgoing proton orbital angular momenta. Since protons with small values of the orbital angular momentum, l_p, come into the range of the nuclear interaction, the expansion coefficients of the experimental reaction amplitude for small values of l_p are in general different from those of the Butler reaction amplitude. For large values of l_p, however, we can expect that the ratio of the coefficients of the expansion of the experimental reaction amplitude to the corresponding ones of the Butler amplitude is constant beyond some minimum value of l_p, because protons with sufficiently large value of l_p pass outside the nucleus. If this constant ratio is n, we must multiply by n^2 the value of the reduced width obtained by the normalization method in order to get its true value. McELLISTREM [8] has applied Bowcock's method to the $C^{12}(d, p)C^{13*}$ (3.086 MeV state) at the incident deuteron energy of 2.889 MeV, and found that for $l_p \geq 4$ the two sets of the expansion coefficients have a constant ratio $n = 1/1.38$.

The value of the reduced width obtained by the normalization method must thus be divided by $(1.38)^2 = 1.9$.

In order to compare the method of extrapolation with the method of Bowcock we have applied the method of extrapolation to this case [9]. The extrapolation length in this case is somewhat large and the pole is at $\cos \theta_p = 1.38$, where θ_p is the center-of-mass angle of the outgoing proton. The extrapolation gives the value of the reduced width 2.4 times smaller than the one obtained by the normalization method. One sees that the extrapolation method and Bowcock's method give values of the reduced width consistent with each other, as is expected.

Since for $\cos \theta > 1$ Legendre polynomials with large value of l_p dominate, near the pole, terms of high l_p value in the expansion of the reaction amplitude contribute more than those of small l_p value, and one may expect that the extrapolation of the terms with $l_p \geq 4$ in the expansion of the reaction amplitudes gives the same value of the reduced width as the one deduced from the extrapolation of the angular distribution. In fact, we have extrapolated the experimental and the Butler theoretical reaction amplitudes with $l_p \geq 4$ given by McELLISTREM [8] for the case treated above, and found that the value of the reduced width so obtained is only 1.3 times smaller than the one obtained by the extrapolation of the angular distribution.

It is interesting to note that this situation in the deuteron stripping reaction has some resemblance to the case of the two-nucleon potential. According to

[8] M. T. McELLISTREM: *Phys. Rev.*, **111**, 596 (1958).

[9] M. NAGASAKI: *XLVI Congresso Nazionale di Fisica* (Napoli, 1960).

TAKETANI [10], the one-pion exchange potential, that is the two-nucleon potential due to the Born approximation (with renormalized coupling constant), can represent the interaction between two nucleons only for their mutual separation greater than the Compton wave length of the pion. In the recent dispersion-theoretical approach [11], this is expressed as the nearest pole in the unphysical region of the momentum transfer. Recently, FURUICHI and MACHIDA [12] have applied the concept of the impact parameter to the dispersion-theoretical approach in order to bring the methodology due to TAKETANI into the dispersion-theoretical approach.

In our case, the virtual momentum (divided by $\hbar$) of the proton in the incident deuteron is of the order of, or less than $\frac{1}{2}k_d + \alpha$ which takes the value $0.90 \cdot 10^{13}$ cm^{-1} in the case of $C^{12}(d, p)C^{13*}$ treated above. If we take $4.7 \cdot 10^{-13}$ cm for a as is done in ref. (8) we have $(\frac{1}{2}k_d + \alpha)a = 4.2$. We can thus expect that protons with orbital angular momentum less than or nearly equal to 4 come into or near to the nuclear surface and are scattered by the nucleus, while protons with orbital angular momentum greater than 4 are not scattered by the nucleus, in good accordance with the minimum value of l_p in the analysis of the reaction with Bowcock's method beyond which the ratio of the coefficients of the expansion of the experimental reaction amplitude to the corresponding ones of the Butler amplitude is found to be constant.

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[10] M. TAKETANI, S. NAKAMURA and M. SASAKI: *Prog. Theor. Phys.*, **6**, 581 (1951). See also *Prog. Theor. Phys. Suppl.*, No. **3**, (1956).
[11] S. FURUICHI and S. MACHIDA: *Nuovo Cimento*, **19**, 396 (1961).
[12] See for example, G. F. CHEW: *Ann. Rev. Nucl. Sci.*, **9**, 29 (1959).

RIASSUNTO

È stata verificata la equivalenza dell'approssimazione di Butler e quella di Born per la reazione di stripping del deuterio nel metodo di continuazione di determinare la larghezza ridotta del nucleone catturato, ed è stata discussa la relazione fra il metodo di continuazione e quello di Bowcok. Il paragone numerico è dato per il caso particolare di $C^{12}(d, p) C^{13*}$ (lo stato di 3.09 MeV) ad $E_d = 2.89$ MeV.

Local Fields with Terminating Expansions (*).

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Summary. — The class of field theories in which the current admits of a terminating expansion in a free field is studied. It is shown that, if one imposes the condition of locality, the current in question reduces to a local polynomial and the resulting field has the trivial S-matrix unity.

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1. – One of the challenging problems in the quantum theory of fields is to construct explicitly a local Lorentz invariant model field theory which has an asymptotic particle interpretation and gives rise to an S-matrix different from unity. Such a model, if it existed, would definitely prove that it is possible to have an interacting dynamical system within the framework of the axiomatic theory. Unfortunately, all the known examples of consistent field theories so far have the trivial S-matrix $S = 1$. The examples known to date fall into two groups. The first one is the so called generalized Wightman polynomial [1],

$$(1) \qquad\qquad \psi(x) = \sum_{i=1}^{n} a_i : \varphi^i(x) : \, ,$$

where φ is a free field and the double dot denotes the normal product [2]; the

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[1] A. S. Wightman: *Problèmes mathématiques de la théorie quantique des champs*, University of Paris lecture notes (1957-58; unpublished).
[2] G. C. Wick: *Phys. Rev.*, **80**, 268 (1950).

second known example is the generalized free field ([3,4]),

$$(2) \qquad A(x) = \int_0^\infty d\mu^2 \varrho(\mu^2)\, \varphi(x, \mu)\,.$$

Here μ denotes the mass of the free field φ in question, and ϱ is an appropriate non-negative measure.

It can be shown for both models that $S \equiv 1$ identically. A way to see this easily is to notice that both $\psi(x)$ and $A(x)$ are symmetrical with respect to past and future, and the asymptotic limits for both $t \to + \infty$ and $t \to - \infty$ coincide with $\varphi(x)$.

All attempts to generalize these two models so as to include interaction have thus far failed. A suggestion of Wightman's was to take (1) as a current and obtain a local solution of the equation:

$$(3) \qquad (\square^2 + \mu^2)\, B(x) = \sum_1^n a_i : \varphi^i(x): = J(x)\,,$$

$B(x)$ can be expected to have a non-trivial S-matrix in general. WIGHTMAN ([5]) has, however, shown that, in the case the current has the specific form $J(x) = = g:\varphi^2(x):$, the resulting $B(x)$ is not local. In the case of the more general $J(x)$ given by (1), ARAKI, HAAG and SCHROER ([6]) have proved that if $J(x)$ is complete, the resulting S-matrix is identically equal to unity. Here a complete field is defined by the property that the only operators that commute with arbitrary polynomials in the field in question are scalar multiples of the unit operator. If $J(x)$ is incomplete, the same authors have shown that the two-particle scattering amplitude must vanish, again emphasizing the difficulties of constructing a field with a non-trivial S-matrix.

It is the purpose of this paper to extend and partially generalize the above mentioned results, also with mainly negative conclusions. In what follows, we will restrict ourselves exclusively to the case where $J(x)$ is complete, as the case of incomplete J is treated in detail in the paper of reference ([6]) and is disposed of as being unphysical. We will take as our model a field whose

([3]) O. W. GREENBERG: *Bull. Am. Phys. Soc.*, **6**, 306 (1961), Washington Meeting.

([4]) G. F. DELL'ANTONIO: *On the support of a field in p-space*, Northwestern University preprint. We are indebted to Dr. DELL'ANTONIO for a discussion on these points.

([5]) A. S. WIGHTMAN and S. T. EPSTEIN: *Ann. Phys.*, **11**, 201 (1960).

([6]) H. ARAKI, R. HAAG and B. SCHROER: *Nuovo Cimento*, **19**, 90 (1961).

current admits of the following expansion:

$$(4) \qquad J(x) = \sum_{K=2}^{N} \sum_{n+m=K} \int \dots \int \mathrm{d}^4 p_1 \dots \mathrm{d}^4 p_n \, \mathrm{d}^4 q_1 \dots \mathrm{d}^4 q_m \cdot$$

$$\cdot \delta(p_1^2 - \mu^2) \dots \delta(q_m^2 - \mu^2) \theta(p_1^0) \dots \theta(q_m^0) \exp\left[ix(p_1 + \dots + p_n - q_1 \dots - q_m) \right] \cdot$$

$$\cdot f(q_1 \dots q_m \mid p_1 \dots p_n) a^+(q_1) \dots a^+(q_m) a(p_1) \dots a(p_n) \,.$$

Here the particle operators are defined in terms of the free field $\varphi(x)$ in the usual manner:

$$(2\pi)^4 \, a(p) \, \theta(p_0) \, \delta(p^2 - \mu^2) = \int \mathrm{d}^4 x \, \exp\left[- ip \cdot x \right] \varphi(x) \,.$$

and similarly for a. The quantities $f(q_1 \dots q_m \mid p_1 \dots p_n)$ are Lorentz invariant functions of the vectors in question and also are symmetric in $q_1 \dots q_m$ and $p_1 \dots p_n$ separately. They are further assumed to be sufficiently smooth and continuous and to be bounded by a polynomial at infinity. Apart from these restrictions, they can be arbitrary. It is to be noted that any field for which one of the asymptotic fields is complete admits of a similar expansion [7] in terms of that asymptotic field, with the important difference that the expansion is not in general required to terminate at a finite order. Furthermore, here we do not necessarily identify the free field $\varphi(x)$ with any of the asymptotic fields.

At this point it is convenient to *define* the set of fields, which we denote by $D(x)$, by the following conditions:

(5)
- a) $D(x)$ admits a finite expansion of the form given by (4).
- b) The functions $f(q_1 \dots q_m \mid p_1 \dots p_n)$ appearing in this expansion are finite polynomials in the scalar products of the argument vectors.
- c) These functions further satisfy
  $$n! \, m! \, f(q_1 \dots q_m \mid p_1 \dots p_n) = (n-1)! \, (m+1)! \, f(q_1 \dots q_m(-p_1) \mid p_2 \dots p_2) \,.$$

It is easy to see that when the f's are all constants, $D(x)$ reduces to the Wightman polynomial given by (1). In the general case, $D(x)$ is a local polynomial constructed not only out of the free field $\varphi(x)$ but also the space derivatives thereof; and is therefore a straightforward extension of (1).

All the results proved in reference [6] carry over without any difficulty to the case where the current is given by this slightly more general form.

[7] R. HAAG: *Kgl. Danske Vid. Selskab*, **29**, 12 (1955).

2. – The main content of this paper will be the proof of the following result: *If a local covariant field $H(x)$ has a complete current given by (4), the S-matrix associated with this field is identically equal to unity.* The essential point of the proof consists in showing that a local $J(x)$ given by (4) must necessarily reduce to the local polynomial $D(x)$ defined above. Since $J(x)$ is then local relative to the free field $\varphi(x)$ and $H(x)$ is local with respect to $J(x)$, if $J(x)$ is assumed to be complete, it will follow that $H(x)$ has the trivial S-matrix unity by the use of Borchers theorem [8], exactly as in reference [6].

We now proceed to prove the main lemma: If $J(x)$ is given by (4) and if $[J(x), J(y)] = 0$ for $(x - y)^2 < 0$, then $J(x)$ must necessarily reduce to the local polynomial defined in (5). To this end, we need the following auxiliary result. Let $\psi(x)$ be a local covariant field and define

$$(6) \qquad T(t, \alpha^2, \boldsymbol{\alpha} \cdot \boldsymbol{P}, \boldsymbol{\alpha} \cdot \boldsymbol{Q}, P, Q) = \int d^3r \exp\left[i\boldsymbol{\alpha} \cdot \boldsymbol{r}\right] \langle Q | [J(0), J(x)] | P \rangle .$$

Here Q stands for the many particle state $Q \equiv (q_1, ..., q_m)$ and $P \equiv (p_1, ..., p_n)$ with $\boldsymbol{\alpha} \cdot \boldsymbol{P}$ denoting the family of variables $\boldsymbol{\alpha} \cdot \boldsymbol{p}_1, ..., \boldsymbol{\alpha} \cdot \boldsymbol{p}_n$ and likewise for $\alpha \cdot \boldsymbol{Q}$. $\boldsymbol{r}$ and t represent the space and time components of x in a fixed frame of reference, and $\boldsymbol{\alpha}$ is a three dimensional vector in the same frame. The particular dependence of T on $\boldsymbol{\alpha}$ is required by ordinary rotation invariance. Now if T is given by this expression and exists for real $\boldsymbol{\alpha}$, it must be an entire function of the vector $\boldsymbol{\alpha}$ in the complex plane, and further the set of functions $\partial^h T / \partial t^h$ must reduce to polynomials in α^2, $\boldsymbol{\alpha} \cdot \boldsymbol{P}$ and $\boldsymbol{\alpha} \cdot \boldsymbol{Q}$ for $t = 0$.

To prove the first part of the statement, we observe that since ψ is a local field, the domain of integration over r is restricted by the inequality $|r| < |t|$, hence $\boldsymbol{\alpha}$ can be continued to any arbitrary complex vector for fixed t. The second statement follows from the fact that for $t = 0$, the integrand has support only at $r = 0$, and therefore reduces to the superposition of a δ-function and finite order derivatives thereof.

Let us now take $J(x)$ given by (6) for the local field in question, and also take $Q = 0$, $P \equiv (p_1, ..., p_{2(N-1)})$, where the particle states are defined in terms of the particle operators of $\varphi(x)$. This is easily seen to be the highest non-vanishing matrix element. Evaluating the expression in question explicitly in terms of the expansion functions f, we get

$$(7) \qquad T(t, \alpha^2, \boldsymbol{\alpha} \cdot \boldsymbol{p}_1, ..., \boldsymbol{\alpha} \cdot \boldsymbol{p}_L, P_1, ... P_L) =$$

$$= \sum_{(v)} \frac{C_{(v)}}{\mathscr{K}^0_{(v)}} \exp\left[it(\sum P^0_{(v)} + \mathscr{K}^0_{(v)})\right] f(\mathscr{K}_{(v)} | P_{(v)}) f(|\mathscr{K}_{(v)} P_{(v)}) + \quad \cdots$$

$$+ \frac{C'_{(v)}}{\mathscr{K}^0_{(v)}} \exp\left[it(\sum P^0_{(v)} - \mathscr{K}_{(v)})\right] J(|\mathscr{K}'_{(v)} P_{(v)}) f(\mathscr{K}'_{(v)} | P_{(v)}).$$

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[8] H. J. BORCHERS: *Nuovo Cimento*, **15**, 784 (1960).

Here $L = 2(N-1)$ and the complexion (L) stands for the set of indices $(1, 2, ..., L)$, (U) is a proper subset of (L) and (V) is the complement of (U) with respect to (L). The notation $\sum_{(U)}$ indicates that the summation is to be carried over all such subsets (U), and $\sum P^0_{(U)}$ denotes the sum of all p^0's with indices belonging to (U). The four-vectors denoted by $\mathscr{K}$ are defined by the equation:

$$(8) \quad \begin{cases} \mathscr{K}^0_{(U)} > 0 , \quad \mathscr{K}^2_{(U)} = \mu^2 , \quad \mathscr{K}^{0'}_{(U)} > 0 , \quad \mathscr{K}'^2_{(U)} = \mu^2 , \quad \mathscr{K}_{(U)} + \boldsymbol{\alpha} + \sum \boldsymbol{P}_{(U)} = 0 . \\ -\mathscr{K}'_{(U)} + \boldsymbol{\alpha} + \sum \boldsymbol{P}_{(U)} = 0 , \quad \mathscr{K}^0_{(U)} = \mathscr{K}^{0'}_{(U)} . \end{cases}$$

The C's are simple numerical constants which are irrelevant for our purposes.

The first point to observe about this equation is that from the assumed existence and proper behaviour of the f-function, it follows that T and all the derivatives of T with respect to time exist. The second point is that there is no integration left on the right side of (7); therefore, if we consider this equation together with all the other equations obtained by differentiating both sides with respect to time, they form a linear set of equations for the particular combinations of f-functions appearing in terms of T. Now consider the set of equations:

$$(9) \quad \left(\frac{\partial^h T}{\partial t^h}\right)_{t=0} = \sum_{(U)} \frac{C_{(U)}}{\mathscr{K}^0_{(U)}} i^h \left(\sum P^0_{(U)} + \mathscr{K}^0_{(U)}\right)^h \cdot$$

$$\cdot f(\mathscr{K}_{(U)} \mid P_{(U)}) f(\cdot \mathscr{K}_{(U)}, P_{(U)}) + \frac{C'_{(U)}}{\mathscr{K}^{0'}_{(U)}} i^h \left(\sum P^0_{(U)} - K^{0'}_{(U)}\right)^h f(\cdot \mathscr{K}'_{(U)}, P_{(U)}) f(\mathscr{K}'_{(U)}, P_{(U)}) .$$

If f's are considered as the unknown quantities, the determinant of the above set of linear equations clearly does not vanish identically, and therefore one can solve them in terms of $(\partial^h T/\partial t^h)_{t=0}$. Using the fact that $(\partial^h T/\partial t^h)_{t=0}$ is a polynomial in α^2, $\boldsymbol{\alpha} \cdot \boldsymbol{p}_1, ..., \boldsymbol{\alpha} \cdot \boldsymbol{p}_L$, we easily get

$$(10) \quad f(\mathscr{K}_{(U)} \mid P_{(V)}) f(\mid \mathscr{K}_{(U)} P_{(U)}) = R(\alpha^2, \boldsymbol{\alpha} \cdot \boldsymbol{p}_1, ..., \boldsymbol{\alpha} \cdot \boldsymbol{p}_L; p_1, ..., p_L) .$$

Here R is a rational function of the first set of variables separated by a semicolon from the rest, and (U) is supposed to run over all possible configuration of indices. We now want to prove that R is actually a polynomial in the indicated variables. To this end, we rewrite eq. (7) in a slightly different form:

$$(11) \quad T = \sum_{(U)} \exp\left[it\sum P^0_{(U)}\right] \left[\cos\{t\mathscr{K}^0_{(U)}\} F_{(U)} + \frac{\sin\{t\mathscr{K}^0_{(U)}\}}{\mathscr{K}^0_{(U)}} F'_{(U)}\right],$$

where

$$F_{(v)} = \frac{1}{\mathcal{K}^0_{(v)}} \left[C_{(v)} f(\mathcal{K}_{(v)} | P_{(v)}) f(|\mathcal{K}_{(v)} P_{(v)}) + C'_{(v)} f(|\mathcal{K}'_{(v)} P_{(v)}) f(\mathcal{K}'_{(v)} | P_{(v)}) \right],$$

$$F'_{(v)} = i[C_{(v)} f(\mathcal{K}_{(v)} | P_{(v)}) f(|\mathcal{K}_{(v)} P_{(v)}) - C'_{(v)} f(|\mathcal{K}'_{(v)} P_{(v)}) f(\mathcal{K}'_{(v)} | P_{(v)})] .$$

Because of their definition, F's also are rational functions in the variables α^2, $\boldsymbol{\alpha} \cdot \boldsymbol{p}_1, ..., \boldsymbol{\alpha} \cdot \boldsymbol{p}_L$, $\mathcal{K}^0_{(v)}$. It is also easy to verify that F and F' are even functions of $\mathcal{K}^0_{(v)}$ and therefore the square root that arises from the definition of $\mathcal{K}^0_{(v)}$ must disappear. Hence, we can conclude that F and F' are rational functions of the quantities α^2, $\boldsymbol{\alpha} \cdot \boldsymbol{p}_1, ..., \boldsymbol{\alpha} \cdot \boldsymbol{p}_L$: Now, take one of the components of $\boldsymbol{\alpha}$, say α_x and decompose F and F' in partial fractions with respect to α_x.

(12)
$$\begin{cases} F_{(v)} = \sum_i \frac{E^i_{(v)}}{(\alpha_x - \gamma_i)^{n_i}} + P_{(v)} , \\ F'_{(v)} = \sum_i \frac{E'^i_{(v)}}{(\alpha_x - \gamma_i)^{n_i}} + P'_{(v)} . \end{cases}$$

Here E's and γ's do not depend on α_x, and P's are polynomials in further, all γ's are required to be distinct. We will now prove that all E's must vanish, leaving only the polynomial. To see this, we use the fact that T is an entire function of $\boldsymbol{\alpha}$, and therefore in the final sum the numerators must combine so as to cancel the denominators. Using the fact that γ's are distinct, this leads to a condition of the following form:

(13)
$$\sum_{(v)} \exp \left[it \sum P^0_{(v)} \right] \left[\cos \{t\mathcal{K}^0_{(v)}\} E^i_{(v)} + \frac{\sin \{t\mathcal{K}^0_{(v)}\}}{K^0_{(v)}} E^{i'}_{(v)} \right] = 0$$

for $\alpha_x = \gamma_i$ for all i.

The $\mathcal{K}$'s are in general distinct since $\boldsymbol{p}$'s are arbitrary. It is easy to see that such an equation cannot be satisfied for arbitrary p's and arbitrary t unless all E's vanish, proving our original assertion. The same argument can be applied to the other components of $\boldsymbol{\alpha}$, with the conclusion that F and F' are polynomials in $\boldsymbol{\alpha}$; or equivalently, in the variables α^2, $\boldsymbol{\alpha} \cdot \boldsymbol{p}_1, ..., \boldsymbol{\alpha} \cdot \boldsymbol{p}_L$.

Solving for the f's in terms of F's, we get

(14) $$f(\mathcal{K}_{(v)} | P_{(v)}) f(|\mathcal{K}_{(v)} P_{(v)}) =$$
$$= Q^{(1)}_{(v)}(\alpha^2, \boldsymbol{\alpha} \cdot \boldsymbol{p}_1, ..., \boldsymbol{\alpha} \cdot \boldsymbol{p}_L; p_1, ..., p_L) + \mathcal{K}^0_{(v)} Q^{(2)}_{(v)}(\alpha^2, \boldsymbol{\alpha} \cdot \boldsymbol{p}_1, ..., \boldsymbol{\alpha} \cdot \boldsymbol{p}_L; p_1, ..., p_L),$$

where Q's are polynomials in the first set of variables.

We now take all the vectors P_i with i belonging to set (V) to lie completely

along the time axis. For this particular configuration of $P_{(V)}$, $f(\mathcal{K}_{(U)}|P_{(V)})$ is only function of $\mathcal{K}^0_{(U)}$ from Lorentz invariance, and we denote it by $1 \cdot \chi(\mathcal{K}^0_{(U)})$. We also switch our independent variable from α to $\mathcal{K}_{(U)}$, clearly without changing the polynomial character of Q's, and also drop the index (U), with the result

(15) $\qquad f(|\mathcal{K} P_{(U)}) = \chi(\mathcal{K}^0)[S_1(\mathcal{K}^2, \boldsymbol{\mathcal{K}} \cdot \boldsymbol{P}_{(U)}; P_{(U)}) + \mathcal{K}^0 S_2(\mathcal{K}^2, \boldsymbol{\mathcal{K}} \cdot \boldsymbol{P}_{(U)}; P_{(U)})]$,

where S_1 and S_2 are polynomials in $\mathcal{K}^2$ and $\boldsymbol{\mathcal{K}} \cdot \boldsymbol{P}_{(U)}$. Since $f(\mathcal{K} P_{(U)})$ is a symmetric function of $\mathcal{K}$ and of each of the P's that constitute the set $P_{(U)}$, it follows that the right hand side of (15) is not only a polynomial in the variables $\boldsymbol{\mathcal{K}} \cdot \boldsymbol{P}_{(U)}$, but also in the variables of the form $\boldsymbol{p}_i \cdot \boldsymbol{p}_j$. At this point, we invoke the Lorentz invariance of f, which requires it to be only a function of the covariant scalar products $\mathcal{K} \cdot P_{(U)}$, $p_i \cdot p_j$, etc. This, combined with the previous statement about the polynomial dependence of f on the three-dimensional scalar products, establishes the result that f is a polynomial in the four-dimensional scalar product $\mathcal{K} \cdot p_i$, $p_i \cdot p_j$, etc.

The same method can be applied to mixed elements by first taking $Q \equiv (p_1)$, $P \equiv (p_2, ..., p_{2(N-1)})$ and continuing in the same fashion. One can then easily establish the result that $f(p_1, ..., p_s|p_{s+1}, ..., p_N)$ is a polynomial in the variables $p_i \cdot p_j$ and also verify the crossing relations given in (5c). We have, therefore, proved that the term of the highest order in the expansion (5) must be a local polynomial in $\varphi(x)$, if $J(x)$ is a local current.

The rest of the proof proceeds by induction. We can now decompose the current as follows:

(16) $\qquad\qquad\qquad J(x) = J_{N-1}(x) + P_N(x)$,

where $P_N(x)$ is the local polynomial containing up to N particle operators, and $J_{N-1}(x)$ has an expansion that ends up with $N-1$ particle operators. We can now consider a matrix element of the commutator between vacuum and a state containing $L-2 = 2N-4$ particles, with the result,

(17) $\quad T(t, \boldsymbol{\alpha}, p_1, ..., p_{L-2}) = \int \mathrm{d}^3 r \exp[i\boldsymbol{\alpha} \cdot \boldsymbol{r}] \langle 0|[J(0), J(x)]|p_1, ..., p_{L-2}\rangle =$

$$= \int \mathrm{d}^3 r \exp[i\boldsymbol{\alpha} \cdot \boldsymbol{r}] \langle 0|[J_{N-1}(0), J_{N-1}(x)]|p_1, ..., p_{L-2}\rangle +$$

$$+ \int \mathrm{d}^3 r \exp[i\boldsymbol{\alpha} \cdot \boldsymbol{r}] \langle 0|[P_N(0), P_N(x)]|p_1, ..., p_{L-2}\rangle .$$

The last term on the right hand side satisfies the same analyticity properties that T satisfies, namely, it is an entire function in α and its derivatives with respect to t reduce to polynomials in α at $t = 0$. These statements can easily be verified using the well-known properties of the local polynomial. Hence

considering the expression

$$T(t, \boldsymbol{\alpha}, p_1, ..., p_{L-2}) - \int \mathrm{d}^3 r \exp\left[i\boldsymbol{\alpha} \cdot \boldsymbol{r}\right] \langle 0 | [P_N(0), P_N(x)] | p_1, ..., p_{L-2} \rangle \,,$$

instead of T, our previous argument can be applied to $J_{N-1}(r)$ unchanged, with identical conclusions. It is now clear that this induction process continued leads to the result that $J(x)$ must be a local polynomial in $\varphi(x)$. Therefore our main lemma is proved and the result stated at the beginning of this paper is established.

3. – The result we have just proved strengthens the negative conclusions of reference ([6]). It shows that in order to get an interaction, the current $J(x)$ is not allowed to have a terminating expansion in terms of a free field. This result is also a natural extension of the theorem that if the S-matrix is different from unity, the outgoing field cannot admit of a terminating expansion in terms of the incoming asymptotic field ([9]). The absence of a finite connection between a free field and an interacting field tends to confirm the belief that perhaps in local quantum field theories with particle interpretation the requirement of the existence of scattering forces the physical states to be such that it is not possible to identify them with a linear space spanned by « free » particle states. This also appears plausible from recent developments in quantum field theory using an indefinite metric, and a definite answer either in the positive or in the negative would be very desirable.

* * *

([9]) O. W. GREENBERG: private communication. We are indebted to Dr. GREENBERG for a discussion of this topic.

RIASSUNTO (*)

Studiamo la classe di teorie di campo in cui la corrente ammette in un campo libero uno sviluppo dotato di limite finito. Mostriamo che, se si impone la condizione di localizzazione, la corrente in questione si riduce ad un polinomio locale ed il campo risultante ha la banale unità della matrice S.

(*) *Traduzione a cura della Redazione.*

Sull'uso degli operatori di proiezione
per ottenere gli elementi di matrice per particelle di spin $\frac{1}{2}$.

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Riassunto. — Allo scopo di dare delle formule per il calcolo immediato in forma relativistica degli elementi di matrice fra stati di particelle libere di spin $\frac{1}{2}$, si ricorre all'uso degli operatori di proiezione. Si ottengono esplicitamente delle espressioni che permettono di calcolare l'elemento di matrice di un qualsiasi operatore a meno di una fase.

Introduzione.

Per calcolare l'elemento di matrice

$$\overline{u}_2 O u_1$$

di un operatore O tra gli stati iniziale e finale di una particella a spin $\frac{1}{2}$ libera, rappresentati dagli spinori a quattro componenti u_1 e u_2, lo considereremo come una traccia

$$\mathrm{tr}\,(O\,u_1\overline{u}_2)$$

e verrà mostrato che la matrice a quattro righe e quattro colonne $u_1\overline{u}_2$ può essere scritta nella forma

$$u_1 u_2 = \frac{P_1 P_2 \varepsilon_1 \exp[i\varphi_{12}]}{[\varepsilon_1\varepsilon_2\,\mathrm{tr}\,(P_1 P_2)]^{\frac{1}{2}}}\,.$$

P_1 e P_2 essendo gli operatori di proiezione (1) per gli stati u_1 e u_2 rispettiva-

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(1) Per il metodo degli operatori di proiezione vedasi anche F. I. FEDOROV: *Žurn. Éksp. Teor. Fiz.*, **30**, 493 (1958).

mente ed ε_1 e ε_2 assumendo i valori $+1$ o -1 secondo il segno dell'energia corrispondente ai due stati. φ_{12} è zero se u_1 ed u_2 coincidono.

Eseguendo i quadrati di elementi di matrice il fattore di fase scompare a causa della identità

$$\exp\left[i(\varphi_{12} + \varphi_{21})\right] = \varepsilon_1\varepsilon_2 \, .$$

Si ottiene allora il risultato ordinario, in quanto

$$\frac{\operatorname{tr}\,(R'(P_2 P_1)\,\operatorname{tr}\,(Q P_1 P_2)}{\varepsilon_1\varepsilon_2\,\operatorname{tr}\,(P_1 P_2)} = \varepsilon_1\varepsilon_2\,\operatorname{tr}\,(R' P_2 Q P_1) \, ,$$

a causa dell'identità

$$P_2 Q P_1 = \frac{\operatorname{tr}\,(P_2 Q P_1)}{\operatorname{tr}\,(P_2 P_1)}\, P_2 P_1 \, .$$

1. – Notazioni e forma esplicita degli operatori di proiezione.

Le notazioni adoperate sono le correnti [2], salvo l'uso dei simboli q_μ per il quadrivettore normalizzato di energia e quantità di moto ($q_0 \lessgtr 0$) e $\mathscr{L}_\mu$ per il quadrivettore di spin.

Per lo stato u_1 si ha così

$$(1) \qquad \underset{1}{q}_\alpha \underset{1}{q}^\alpha = 1 \, , \qquad \underset{1}{\mathscr{L}}_\alpha \underset{1}{\mathscr{L}}^\lambda = -1 \, , \qquad \underset{1}{q}_\alpha \underset{1}{\mathscr{L}}^\lambda = 0$$

e l'equazione di Dirac si scrive

$$(2) \qquad (\underset{1}{q}_\mu \gamma^\mu - I)u_1 = 0 \, .$$

L'operatore di proiezione P_1 è

$$(3) \quad P_1 = \tfrac{1}{4}(\underset{1}{q}_\mu \gamma^\mu + I)(i\underset{1}{\mathscr{L}}_\nu \gamma^\nu \gamma_5 + I) = \tfrac{1}{4}(I + \underset{1}{q}_\mu \gamma^\mu + i\underset{1}{\mathscr{L}}_\nu \gamma^\nu \gamma_5 + i\underset{1}{q}_\mu \underset{1}{\mathscr{L}}_\nu \gamma^\mu \gamma^\nu \gamma_5) \, ,$$

e $P_1^2 = P_1$. I due fattori in P_1 commutano. Nella somma $\underset{1}{q}_\mu \underset{1}{\mathscr{L}}_\nu \gamma^\mu \gamma^\nu \gamma_5$ deve intendersi, come regola generale (vedi Appendice), $\mu \neq \nu$.

Un'espressione analoga vale per P_2.

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[2] Si veda ad es. il libro: S. S. SCHWEBER, H. A. BETHE e F. DE HOFFMANN: *Mesons and Fields* (Evanston, 1955). La metrica usata è $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$ ed abbiamo $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I$, $\gamma_\mu = g_{\mu\lambda}\gamma^\alpha$ e $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$.

Useremo anche le notazioni

$$
(4) \quad
\begin{cases}
s_\mu = q_{1\mu} + q_{2\mu} & S_\mu = \mathcal{L}_{1\mu} + \mathcal{L}_{2\mu} , \\
d_\mu = q_{1\mu} - q_{2\mu} . & D_\mu = \mathcal{L}_{1\mu} - \mathcal{L}_{2\mu} .
\end{cases}
$$

Indicando i quadrivettori con semplici lettere e i prodotti scalari con parentesi tonde, le condizioni (1) diventano

$$
(5) \quad
\begin{cases}
(SS) + (DD) = -4 , & (ss) + (dd) = 4 , \\
(sS) = -(dD) , & (sD) = -(dS) , \qquad (sd) = (SD) = 0 .
\end{cases}
$$

2. – Espressione della matrice $u_1 \bar{u}_2$ tramite gli operatori di proiezione P_1 e P_2.

La matrice a quattro righe e quattro colonne

$$
A = u_1 \bar{u}_2 \qquad\qquad (\bar{u}_2 = u_2^\dagger \gamma^0),
$$

è caratterizzata [3] a meno di un fattore numerico dalle condizioni

$$
(6) \quad
\begin{cases}
P_1 A \gamma^{0\dagger} = A \gamma^{0\dagger} , \\
A \gamma^{0\dagger} P_2^\dagger = A \gamma^{0\dagger} ,
\end{cases}
$$

(è comunque $\gamma^0 \gamma^{0\dagger} = 1$).

Ma ogni matrice $P_1 B P_2^\dagger$ soddisfà queste condizioni e così dobbiamo avere

$$
(7) \qquad\qquad P_1 B P_2^\dagger = K_B^{1,2} u_1 \bar{u}_2 \gamma^{0\dagger} ,
$$

dove il numero complesso $K_B^{1,2}$ potrebbe essere anche zero.

Risulta pure, tenendo conto che

$$
(8) \qquad\qquad \gamma^0 P_2^\dagger \gamma^0 = P_2 ,
$$

l'identità

$$
(9) \qquad\qquad P_1 B P_2 = \frac{\text{tr}\,(P_1 B P_2)}{\text{tr}\,(P_1 P_2)}\, P_1 P_2 .
$$

[3] Si possono sempre trovare quattro stati ortogonali u_k', di cui uno coincidente con u_1, ed altri quattro stati u_j'', pure ortogonali, di cui uno coincida con u_2. Con questi stati si possono costruire sedici matrici linearmente indipendenti a quattro righe e quattro colonne $u_k' \bar{u}_j'' \gamma^{0\dagger}$ tutte soddisfacenti l'identità $P_1 u_k' \bar{u}_j'' \gamma^{0\dagger} P_2^\dagger = 0$ ad eccezione di $u_1 \bar{u}_2 \gamma^{0\dagger}$.

Il modulo di $K_B^{1,2}$ è ottenibile dalla condizione di normalizzazione. Sceglieremo $B = \gamma^0$ e scriveremo $K^{1,2}$ per $K_{,0}^{1,2}$.

Si ha allora

$$(10) \qquad P_1 P_2 = K^{1,2} u_1 \bar{u}_2$$

e la condizione di normalizzazione dà

$$\mathrm{tr}\,(P_1 P_2 P_2 P_1) = K^{1,2} K^{2,1} \,\mathrm{tr}\,(u_1 \bar{u}_2 u_2 \bar{u}_1) = K^{1,2} K^{2,1} \varepsilon_1 \varepsilon_2 \;.$$

Ma

$$\mathrm{tr}\,(P_1 P_2 P_2 P_1) = \mathrm{tr}\,(P_1 P_2) = K^{1,2} \bar{u}_2 u_1 = K^{2,1} \bar{u}_1 u_2$$

e dunque

$$(11) \qquad \bar{u}_2 u_1 = K^{2,1} \varepsilon_1 \varepsilon_2 \qquad \text{e} \qquad \bar{u}_1 u_2 = K^{1,2} \varepsilon_1 \varepsilon_2 \;.$$

Supponendo $(\gamma^0)^2 = g^{00} = 1$, si ha $\gamma^{0\dagger} = \gamma^0$ e

$$(12) \qquad \bar{u}_2 u_1 = (\bar{u}_1 u_2)^* \;.$$

Questo porta a $K^{1,2} = (K^{2,1})^*$ ed a

$$(13) \qquad \mathrm{tr}\,(P_1 P_2) \varepsilon_1 \varepsilon_2 = (\bar{u}_1 u_2)(\bar{u}_2 u_1) = |K^{1,2}|^2 = |K^{2,1}|^2 \geqslant 0 \;.$$

Possiamo perciò assumere

$$(14) \qquad K^{1,2} = [\mathrm{tr}\,(P_1 P_2) \varepsilon_1 \varepsilon_2]^{\frac{1}{2}} \varepsilon_1 \exp[-i\varphi_{12}]$$

e, se $u_1 = u_2$, φ_{12} è zero grazie al fattore ε_1 (usando ε_2 si avrebbe avuto un diverso significato di φ_{12}).

Per la (12) è anche

$$(15) \qquad \exp[i(\varphi_{12} + \varphi_{21})] = \varepsilon_1 \varepsilon_2 \;.$$

Se $\mathrm{tr}\,(P_1 P_2) \neq 0$, quindi

$$(16) \qquad u_1 \dot{u}_2 = \frac{P_1 P_2 \varepsilon_1 \exp[i\varphi_{12}]}{[\mathrm{tr}\,(P_1 P_2) \varepsilon_1 \varepsilon_2]^{\frac{1}{2}}} \;,$$

e

$$(17) \qquad \bar{u}_2 O \dot{u}_1 = \frac{\mathrm{tr}\,(O P_1 P_2) \varepsilon_1 \exp[i\varphi_{12}]}{[\mathrm{tr}\,(P_1 P_2) \varepsilon_1 \varepsilon_2]^{\frac{1}{2}}} \;.$$

$P_1 P_2$ si può scrivere

(18) $P_1 P_2 = \frac{1}{16}\{ +\frac{1}{4}[\mathscr{S}]I + \frac{1}{4}(A_\mu + iB_\mu)\gamma^\mu + \frac{1}{4}(U_{\mu\nu} + iV_{\mu\nu})\gamma^\mu\gamma^\nu +$

$$+ \frac{1}{4}(C_\mu + iE_\mu)\gamma^\mu\gamma_5 + \frac{1}{4}(F + iG)\gamma_5\}\,.$$

Nel calcolo della traccia $\mathrm{tr}\,(O P_1 P_2)$ interviene solo quella parte di $P_1 P_2$ che ha lo stesso carattere tensoriale di O. Si hanno i seguenti risultati

(19)
$$\left|\begin{array}{lll}
O = RI & \mathrm{tr}\,(O P_1 P_2) = & \frac{1}{16}R[\mathscr{S}] \\[4pt]
\;= R_\mu \gamma^\mu & = & \frac{1}{16}(A_\mu + iB_\mu)R^\mu \\[4pt]
\;= R_{\mu\nu}\gamma^\mu\gamma^\nu & = -\frac{1}{16}(U_{\mu\nu} + iV_{\mu\nu})(R^{\mu\nu} - R^{\nu\mu}) \\[4pt]
\;= R_\mu \gamma^\mu\gamma_5 & = & \frac{1}{16}(C_\mu + iE_\mu)R^\mu \\[4pt]
\;= R\gamma_5 & = -\frac{1}{16}(F + iG)R\,.
\end{array}\right.$$

I coefficienti valgono

(20)
$$\left|\begin{array}{l}
[\mathscr{S}] = -(ss)(SS) + (sS)^2 - (sD)^2 \\[6pt]
A_\mu = -2(SS)s_\mu + 2(sS)S_\mu - 2(sD)D_\mu \\[6pt]
B^\mu = -2s_\alpha d_\beta S_\gamma \varepsilon^{\alpha\beta\gamma\mu} \\[6pt]
U_{\mu\nu} \equiv (SS)s_\mu d_\nu + (ss)S_\mu D_\nu + (sS)[-s_\mu D_\nu + d_\mu S_\nu] + (Ds)[s_\mu S_\nu - d_\mu D_\nu] \\[6pt]
V^{\mu\nu} = 2s_\alpha S_\beta \varepsilon^{\alpha\beta\mu\nu} \\[6pt]
C^\mu = -2s_\alpha S_\beta D_\gamma \varepsilon^{\alpha\beta\gamma\mu} \\[6pt]
E_\mu = 2(ss)S_\mu - 2(sS)s_\mu - 2(sD)d_\mu \\[6pt]
F = -s_\alpha d_\beta S_\gamma D_\delta \varepsilon^{\alpha\beta\gamma\delta} = -4q_{1\alpha} q_{2\beta} \mathscr{L}_{1\gamma} \mathscr{L}_{2\delta} \varepsilon^{\alpha\beta\gamma\delta} = -4\mathscr{E} \\[6pt]
G = -4(sD)\,.
\end{array}\right.$$

$P_2 P_1$ può ottenenrsi da $P_1 P_2$ cambiando d in $-d$ e D in $-D$. Si noti che in $P_2 P_1$ rispetto al complesso coniugato di $P_1 P_2$ cambiano i segni dei termini tensoriali e pseudovettoriali. Infatti l'identità $(\bar{u}_1 R' u_2) = (\bar{u}_2 R u_1)^*$, con $R' = \gamma^0 R^\dagger \gamma^0$, può essere scritta $\mathrm{tr}\,(R' P_2 P_1) = \mathrm{tr}\,(R P_1 P_2)^*$, ed il cambio di segno osservato è compensato dall'inversione dell'ordine dei γ in R'.

Come funzioni dei q e degli $\mathscr{L}$ i coefficienti precedenti si scrivono

$$(21) \quad \begin{cases} [\mathscr{S}] = & 4[1 + (\underset{12}{qq}) - (\underset{1}{\mathscr{L}}\underset{2}{\mathscr{L}}) - (\underset{12}{qq})(\underset{1}{\mathscr{L}}\underset{2}{\mathscr{L}}) + (\underset{12}{q\mathscr{L}})(\underset{21}{q\mathscr{L}})] \\[2mm] A_\mu = & 4(\underset{1}{q}_\mu + \underset{2}{q}_\mu)[1 - (\underset{1}{\mathscr{L}}\underset{2}{\mathscr{L}})] + 4(\underset{12}{q\mathscr{L}})\underset{1}{\mathscr{L}}_\mu + 4(\underset{21}{q\mathscr{L}})\underset{2}{\mathscr{L}}_\mu \\[2mm] B^\mu = & 4\underset{1}{q}_\gamma\underset{2}{q}_\beta(\underset{1}{\mathscr{L}}_\gamma + \underset{2}{\mathscr{L}}_\gamma)\,\varepsilon^{\alpha\beta\gamma\mu} \\[2mm] U_{\mu\nu} = & 4\underset{1}{q}_\mu\underset{2}{q}_\nu[1 - (\underset{1}{\mathscr{L}}\underset{2}{\mathscr{L}})] - 4\underset{1}{\mathscr{L}}_\mu\underset{2}{\mathscr{L}}_\nu[1 + (\underset{12}{qq})] - 4(\underset{1}{q}\cdot\underset{2}{\mathscr{L}})\underset{2}{q}_\mu\underset{1}{\mathscr{L}}_\nu + 4(\underset{21}{q\mathscr{L}})\underset{1}{q}_\mu\underset{2}{\mathscr{L}}_\nu \\[2mm] V^{\mu\nu} = & 2(\underset{1}{q}_\alpha + \underset{2}{q}_\alpha)(\underset{1}{\mathscr{L}}_\beta + \underset{2}{\mathscr{L}})\,\varepsilon^{\alpha\beta\mu\nu} \\[2mm] C^\mu = & 4(\underset{1}{q}_\alpha + \underset{2}{q}_\lambda)\underset{1}{\mathscr{L}}_\beta\underset{2}{\mathscr{L}}_\gamma\,\varepsilon^{\alpha\beta\gamma\mu} \\[2mm] E_\mu = & 4(\underset{1}{\mathscr{L}}_\mu + \underset{2}{\mathscr{L}}_\mu)[1 + (\underset{12}{qq})] - 4(\underset{12}{q\mathscr{L}})\underset{2}{q}_\mu - 4(\underset{21}{q\mathscr{L}})\underset{1}{q}_\mu \\[2mm] F = & -4\underset{1}{q}_\gamma\underset{2}{q}_\beta\underset{1}{\mathscr{L}}_\gamma\underset{2}{\mathscr{L}}_\delta\,\varepsilon^{\alpha\beta\gamma\delta} \\[2mm] G = & 4(\underset{12}{q\mathscr{L}}) - 4(\underset{21}{q\mathscr{L}})\,. \end{cases}$$

Da notare anche la seguente proprietà del fattore di fase $\exp[iq_{12}]$

$$\operatorname{tr}[(u_1\bar u_2)(u_2\bar u_3)(u_3\bar u_1)] = \varepsilon_1\varepsilon_2\varepsilon_3 =$$

$$= \varepsilon_1\varepsilon_2\varepsilon_3 \frac{\operatorname{tr}(P_1P_2P_3)\exp[i(\varphi_{12}+\varphi_{23}+\varphi_{31})]}{[\varepsilon_1\varepsilon_2\operatorname{tr}(P_1P_2)]^{\frac12}[\varepsilon_2\varepsilon_3\operatorname{tr}(P_2P_3)]^{\frac12}[\varepsilon_3\varepsilon_1\operatorname{tr}(P_3P_1)]^{\frac12}},$$

cioè

$$(22) \qquad \exp[i(\varphi_{12}+\varphi_{23}+\varphi_{31})] = \frac{[\operatorname{tr}(P_1P_2P_3)]^*}{[\varepsilon_1\varepsilon_2\operatorname{tr}(P_1P_2)]^{\frac12}[\varepsilon_2\varepsilon_3\operatorname{tr}(P_2P_3)]^{\frac12}[\varepsilon_3\varepsilon_1\operatorname{tr}(P_3P_1)]^{\frac12}},$$

che generalizza la (15).

3. – Alcune osservazioni.

Consideriamo P_2QP_1. Poichè

$$(23) \qquad Q\frac{\underset{1}{q}_\mu\gamma^\mu + I}{2} = Q\frac{\underset{1}{q}_\mu\gamma^\mu + I}{2}\frac{\underset{1}{q}_\mu\gamma^\mu + I}{2},$$

si può eseguire la sostituzione

$$(24) \qquad Q \to Q\frac{\underset{1}{q}_\mu\gamma^\mu + I}{2} \qquad \text{cioè} \qquad Q \to Q(\underset{1}{q}_\mu\gamma^\mu)\,.$$

Similmente Q può essere sostituito da

(25) $\qquad Q(i\underset{1}{\mathscr{L}}_\mu\gamma^\mu\gamma_5)\,,\qquad (\underset{2}{q}_\mu\gamma^\mu)Q\,,\qquad (i\underset{2}{\mathscr{L}}_\mu\gamma^\mu\gamma_5)Q\,.$

Questo implica in particolare

(26) $\qquad \underset{1}{q}_\mu\gamma^\mu\to\underset{1}{q}_\mu\gamma^\mu(\underset{1}{q}_\nu\gamma^\nu)=I\,,\quad \underset{2}{q}_\mu\gamma^\mu\to I\,,\quad \underset{1}{\mathscr{L}}_\mu\gamma^\mu\to-i\gamma_5\,,\quad \underset{2}{\mathscr{L}}_\mu\gamma^\mu\to+i\gamma_5\,,$

ovvero

(27) $\qquad s_\mu\gamma^\mu\to 2I\,,\quad d_\mu\gamma^\mu\to 0\,,\quad S_\mu\gamma^\mu\to 0\,,\quad D_\mu\gamma^\mu\to-2i\gamma_5\,.$

Si ottiene pure

(28)
$$\begin{cases}
s_\mu d_\nu\gamma^\mu\gamma^\nu\to(dd)I & s_\mu S_\nu\gamma^\mu\gamma^\nu\to-(sD)I\\[4pt]
s_\mu D_\nu\gamma^\mu\gamma^\nu\to-(sS)I & d_\mu S_\nu\gamma^\mu\gamma^\nu\to(sS)I\\[4pt]
d_\mu D_\nu\gamma^\mu\gamma^\nu\to(sD)I+4i\gamma_5 & S_\mu D_\nu\gamma^\mu\gamma^\nu\to(DD)I\,,
\end{cases}$$

e

(29) $\qquad s_\mu\gamma^\mu\gamma_5\to 0\,,\quad d_\mu\gamma^\mu\gamma_5\to-2\gamma_5\,,\quad S_\mu\gamma^\mu\gamma_5\to-2iI\,,\quad D_\mu\gamma^\mu\gamma_5\to 0\,.$

Ora, se s, d, S, D sono differenti da zero e linearmente indipendenti, ogni operatore può essere espresso tramite i precedenti operatori, così che si può scrivere

(30) $\qquad \mathrm{tr}\,(QP_1P_2)=a\,\mathrm{tr}\,(P_2P_1)+b\,\mathrm{tr}\,(P_2\gamma_5P_1)\,,$

con a e b esprimibili in termini dei vettori duali

$$\underset{s}{F}\,,\quad \underset{d}{F}\,,\quad \underset{S}{F}\,,\quad \underset{D}{F}\,,$$

definiti dalle proprietà, ad esempio per $\underset{s}{F}$,

(31) $\qquad (\underset{s}{F}d)=(\underset{s}{F}S)=(\underset{s}{F}D)=0\,,\qquad (\underset{s}{F}s)=1\,.$

Dunque

(32) $\qquad \underset{s}{F}^\mu=-\dfrac{d_\alpha S_\beta D_\gamma\varepsilon^{\alpha\beta\gamma\mu}}{4\mathscr{E}}\,.$

Per l'identità

(33) $\qquad \delta^\mu{}_\nu=\underset{s}{F}^\mu s_\nu+\underset{d}{F}^\mu d_\nu+\underset{S}{F}^\mu S_\nu+\underset{D}{F}^\mu D_\nu\,,$

ad esempio, si ottiene, se $Q = Q_\mu \gamma^\mu$

$$(34) \qquad Q_\mu = (\underset{s}{F}Q)s_\mu + (\underset{d}{F}Q)d_\mu + (\underset{S}{F}Q)S_\mu + (\underset{D}{F}Q)D_\mu$$

e

$$(35) \qquad \mathrm{tr}\,(QP_1 P_2) = 2\,\mathrm{tr}\,(P_2 P_1)(\underset{s}{F}Q) - 2i\,\mathrm{tr}\,(P_2\gamma_5 P_1)(\underset{D}{F}Q)\,.$$

In generale $P_1 P_2$ se espresso tramite i vettori F assume la forma

$$(36) \qquad \left|\begin{aligned}
A_\mu + iB_\mu &= -2[\mathscr{S}]\,\underset{s}{F}_\mu + 2[4(sD) - 4i\mathscr{E}]\,\underset{D}{F}_\mu \\
U_{\mu\nu} + iV_{\mu\nu} &= [\mathscr{S}](X_{\mu\nu}) + 4[4(sD) - 4i\mathscr{E}]\,\underset{d}{F}_\mu\underset{D}{F}_\nu \\
C_\mu + iE_\mu &= i\{2[\mathscr{S}]\underset{s}{F}_\mu - 2[4(sD) - 4i\mathscr{E}]\,\underset{d}{F}_\mu\} \\
F + iG &= -i[4(sD) - 4i\mathscr{E}]
\end{aligned}\right.$$

con

$$(37) \qquad (X_{\mu\nu}) = (dd)\underset{s}{F}_\mu\underset{d}{F}_\nu + (dS)\underset{s}{F}_\mu\underset{S}{F}_\nu + (qD)\underset{s}{F}_\mu\underset{D}{F}_\nu +$$
$$+ (sD)\underset{d}{F}_\mu\underset{D}{F}_\nu + (sS)\underset{d}{F}_\mu\underset{S}{F}_\nu + (DD)\underset{s}{F}_\mu\underset{D}{F}_\nu\,.$$

Si ha pure

$$(38) \qquad P_2\gamma_5 P_1 = P_2 P_1(-\underset{1}{q}, -\underset{1}{\mathscr{L}})\gamma_5\,,$$

e quindi

$$(39) \qquad \mathrm{tr}\,(P_2\gamma_5 P_1) = i\,\tfrac{1}{16}[4(sD) - 4i\mathscr{E}]\,.$$

L'espressione completa di $P_2\gamma_5 P_1$ è

$$(40) \qquad P_2\gamma_5 P_1 = \tfrac{1}{16}\{\tfrac{1}{4}(\overline{F} - i\overline{G})I + \tfrac{1}{4}(\overline{A}_\mu - i\overline{B}_\mu)\gamma^\mu + \tfrac{1}{4}(-\overline{U}_{\mu\nu} + i\overline{V}_{\mu\nu})\gamma^\mu\gamma^\nu +$$
$$+ \tfrac{1}{4}(-\overline{C}_\mu + i\overline{E}_\mu)\gamma^\mu\gamma_5 + \tfrac{1}{4}[\mathscr{D}]\gamma_5\}\,,$$

dove i coefficienti si ottengono dai seguenti coefficienti di $P_1\gamma_5 P_2$

$$(41) \qquad \left|\begin{aligned}
\overline{F} + i\overline{G} &= -i[4(sD) + i4\mathscr{E}] \\
\overline{A}^\mu + i\overline{B}^\mu &= -2d_\alpha S_\beta D_\gamma\,\varepsilon^{\alpha\beta\gamma\mu} + i[-2(sD)s^\mu - 2(dd)D^\mu - 2(sS)d^\mu] = \\
&= i\{+2[\mathscr{D}]\underset{D}{F}^\mu - 2[4(sD) + i4\mathscr{E}]\underset{s}{F}^\mu\} \\
\overline{U}^{\mu\nu} + i\overline{V}^{\mu\nu} &\equiv \tfrac{1}{2}[(DD)d_\alpha s_\beta + (dd)D_\alpha S_\beta - (sS)(s_\alpha D_\beta - d_\alpha S_\beta) + \\
&\quad + (sD)(s_\alpha S_\beta - d_\alpha D_\beta)]\varepsilon^{\alpha\beta\mu\nu} - i4d^\mu D^\nu \equiv \\
&\equiv \tfrac{1}{2}\{-[\mathscr{D}](Y_{\alpha\beta}) - [4(sD) + i4\mathscr{E}]\,\underset{s}{F}_\alpha\underset{s}{F}_\beta\}\varepsilon^{\beta\mu\nu} \\
\overline{C}^\mu + i\overline{E}^\mu &= -2(DD)d^\mu - 2(sS)D^\mu + 2(sD)S^\mu + i2s_\alpha d_\beta D_\gamma\varepsilon^{\alpha\beta\gamma\mu} = \\
&= +2[\mathscr{D}]\,\underset{d}{F}^\mu - 2[4(sD) + i4\mathscr{E}]\,\underset{s}{F}^\mu
\end{aligned}\right.$$

con

(42) $\quad (Y_{\alpha\beta}) = (ss)F_{\overset{}{s}\alpha}F_{\overset{}{s}\beta} + (sD)F_{\overset{}{s}\alpha}F_{\overset{}{D}\beta} + (sS)F_{\overset{}{s}\alpha}F_{\overset{}{s}\beta} +$

$$+ (dD)F_{\overset{}{s}\alpha}F_{\overset{}{D}\beta} + (dS)F_{\overset{}{s}\alpha}F_{\overset{}{s}\beta} + (SS)F_{\overset{}{D}\alpha}F_{\overset{}{s}\beta} .$$

L'identità

$$P_2\gamma_5 P_1 - \frac{\mathrm{tr}\,(P_2\gamma_5 P_1)}{\mathrm{tr}\,(P_2 P_1)}\, P_2 P_1 ,$$

può essere facilmente controllata.

APPENDICE

Riportiamo qui le regole per l'esecuzione dei prodotti ed altre identità usate nel testo.

Si fa la convenzione che nei prodotti contenenti due o più γ la somma sugli indici sia eseguita con la condizione che essi siano in ogni addendo diversi tra di loro.

Regole per i prodotti di:

due vettori

$$(P_\mu\gamma^\mu)(Q_\nu\gamma^\nu) = (PQ) + P_\mu Q_\nu\gamma^\mu\gamma^\nu ,$$

$$(Q_\mu\gamma^\mu)(P_\nu\gamma^\nu) = (PQ) - P_\mu Q_\nu\gamma^\mu\gamma^\nu ,$$

un vettore ed uno pseudovettore

$$(P_\mu\gamma^\mu)(S_\nu\gamma^\nu\gamma_5) = \quad (PS)\gamma_5 + P_\mu S_\nu\gamma^\mu\gamma^\nu\gamma_5 ,$$

$$(S_\mu\gamma^\mu\gamma_5)(P_\nu\gamma^\nu) = -\,(PS)\gamma_5 + P_\mu S_\nu\gamma^\mu\gamma^\nu\gamma_5 ,$$

due pseudovettori

$$(R_\mu\gamma^\mu\gamma_5)(S_\nu\gamma^\nu\gamma_5) = (RS) + R_\mu S_\nu\gamma^\mu\gamma^\nu ,$$

$$(S_\mu\gamma^\mu\gamma_5)(R_\nu\gamma^\nu\gamma_5) = (RS) - R_\mu S_\nu\gamma^\mu\gamma^\nu ,$$

un vettore ed un tensore

$$(P_\mu\gamma^\mu)\big(T_{\nu\omega}\gamma^\nu\gamma^\omega\{\gamma_5\}\big) = [\quad P^\alpha T_{\alpha\mu}\gamma^\mu - P^\alpha T_{\mu\alpha}\gamma^\mu + P_\mu T_{\nu\omega}\gamma^\mu\gamma^\nu\gamma^\omega]\{\quad\gamma_5\} ,$$

$$\big(T_{\nu\omega}\gamma^\nu\gamma^\omega\{\gamma_5\}\big)(P_\mu\gamma^\mu) = [-\,P^\alpha T_{\alpha\mu}\gamma^\mu + P^\alpha T_{\mu\alpha}\gamma^\mu + P^\mu T_{\nu\omega}\gamma^\mu\gamma^\nu\gamma^\omega]\{-\gamma_5\} ,$$

uno pseudovettore ed un tensore

$$(S_\mu\gamma^\mu\gamma_5)\big(T_{\nu\omega}\gamma^\nu\gamma^\omega\{\gamma_5\}\big) = [\quad S^\alpha T_{\alpha\mu}\gamma^\mu - S^\alpha T_{\mu\alpha}\gamma^\mu + S_\mu T_{\nu\omega}\gamma^\mu\gamma^\nu\gamma^\omega]\gamma_5\{\quad\gamma_5\} ,$$

$$\big(T_{\nu\omega}\gamma^\nu\gamma^\omega\{\gamma_5\}\big)(S_\mu\gamma^\mu\gamma_5) = [-\,S^\alpha T_{\alpha\mu}\gamma^\mu + S^\alpha T_{\mu\alpha}\gamma^\mu + S_\mu T_{\nu\omega}\gamma^\mu\gamma^\nu\gamma^\omega]\gamma_5\{-\gamma_5\} ,$$

due tensori

$$(T_{\mu\nu}\gamma^\mu\gamma^\nu\{\gamma_5\})(U_{\eta\omega}\gamma^\eta\gamma^\omega\{\gamma_5\}) = [T_{\alpha\beta}U^{\beta\alpha} - T_{\alpha\beta}U^{\alpha\beta} + T_{\mu\nu}U_{\eta\omega}\gamma^\mu\gamma^\nu\gamma^\eta\gamma^\omega +$$
$$+ (T_\mu{}^\alpha U_{\alpha\nu} - T_\mu{}^\alpha U_{\nu\alpha} - T^\alpha{}_\mu U_{\alpha\nu} + T^\alpha{}_\mu U_{\nu x})\gamma^\mu\gamma^\nu]\{\gamma_5\}\{\gamma_5\}\,,$$

dove

$$\gamma'_5 = \gamma^0\gamma^1\gamma^c\gamma^3\,,$$
$$\gamma_5^2 = -1\,,$$

$$\varepsilon^{\mu\nu\eta\omega} = 0 \text{ se due indici sono uguali,}$$

$$\varepsilon^{0123} = +1,$$

$$\varepsilon^{\mu\nu\eta\omega} = \pm 1 \text{ secondo la parità della permutazione } (\mu\nu\eta\omega)\,,$$

$$\varepsilon_{\mu\nu\eta\omega} = -\varepsilon^{\mu\nu\eta\omega}.$$

Si ha pure

$$T_{\mu\nu}\gamma^\mu\gamma^\nu\gamma_5 = \tfrac{1}{2}T_{\alpha\beta}\varepsilon^{\alpha\beta\mu\nu}\gamma_\mu\gamma_\nu\,,$$
$$Z_{\mu\nu\eta\omega}\gamma^\mu\gamma^\nu\gamma^\eta\gamma^\omega = Z_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta\gamma\delta}\gamma_5\,,$$
$$W_{\mu\nu\eta}\gamma^\mu\gamma^\nu\gamma^\eta = -W_{\alpha\beta\gamma}\varepsilon^{\alpha\beta\gamma\mu}\gamma_\mu\gamma_5\,,$$

e l'identità

$$\varepsilon^{\mu\nu\eta\omega}A^{\tau*} = \varepsilon^{\tau\nu\eta\omega}A^{\mu*} + \varepsilon^{\mu\tau\eta\omega}A^{\nu*} + \varepsilon^{\mu\nu\tau\omega}A^{\eta*} + \varepsilon^{\mu\nu\eta\tau}A^{\omega*}\,,$$

dove l'asterisco indica un qualunque insieme di indici.

———

SUMMARY

To the purpose of writing down formulae for relativistic immediate computation of matrix elements between free states for a particle of spin $\tfrac{1}{2}$, the use of projection operators is made. Expressions are obtained which allow to compute matrix elements of any operator up to a phase.

Emission of Fast Deuterons and Tritons
from K⁻-Mesons Captured at Rest in Nuclear Emulsion
and a Search for the Hypernucleus ($\Sigma^- n$).

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Summary. — The results of mass measurements on the fast baryons emitted in the capture at rest of 7600 K⁻-mesons are described. The analysis has been restricted to those interactions from which neither charged π-mesons, nor slow Σ-hyperons (kinetic energy $\leqslant 60$ MeV), nor hyperfragments are emitted. As already reported in a previous work (see ref. ([1])), a substantial number of deuterons and tritons is present among the fast baryons: the proportion of deuterons of kinetic energy exceeding 84 MeV is $(11.1\pm2.5)\%$, that of tritons is $(2.3\pm1.0)\%$. Fast deuterons and tritons of ranges greater than 1 cm are emitted from at least $(1\pm0.2)\%$ of K⁻-mesons interactions at rest in nuclear emulsion. Among the fast baryons a number of Σ-hyperons decaying in flight has been observed and measured: no ($\Sigma^- n$) hypernucleus has been detected.

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1. – Introduction.

In the previous work (¹) the emission of energetic deuterons and tritons from K⁻-mesons captured at rest in nuclei of photographic emulsions was reported. Further investigation has substantiated the earlier results. The emission of nuclear fragments of high energies has thus been confirmed and it is reasonable to suppose that if a (Σ^-n) hypernucleus exists it may be emitted with comparable energy. Measurements of ionization and of scattering have been made upon a sample of 36 tracks of fast hyperons in order to detect possible examples of the hypernucleus (Σ^-n). As the measurements required at least 5 mm of track, the time of flight involved ($\sim 10^{-10}$ s) would preclude the possibility of detection of the hypernuclei if they were of very short lifetime. No evidence has been found for the existence of a (Σ^-n) hypernucleus in the present study.

2. – Emission of fast deuterons and tritons.

The emission of fast stable particles (p, d and t) has been investigated in a sample of 7 600 interactions of K⁻-mesons at rest in nuclear emulsion (*). The analysis has been restricted to interactions from
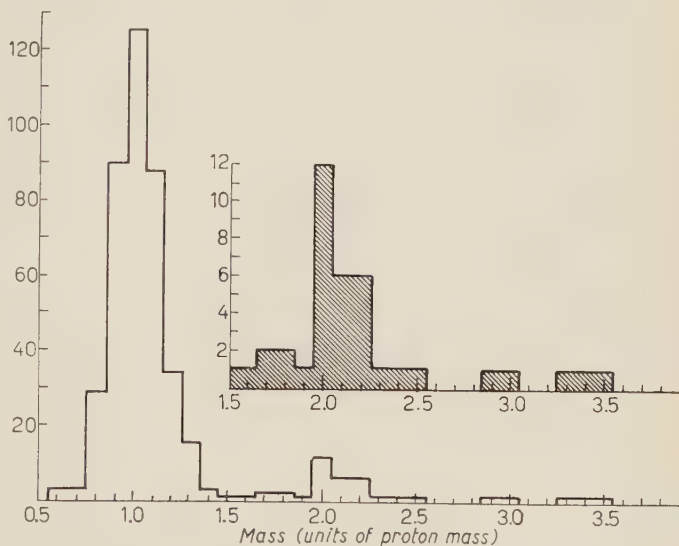
Fig. 1. – Mass spectrum obtained for the protons, deuterons and tritons emitted from K⁻-interactions at rest in emulsion. The inset shows an enlargement of the higher mass end of the spectrum.

(¹) B. D. JONES, B. SANJEEVAIAH, J. ZAKRZEWSKI, P. G. BIZZETTI, J. P. LAGNAUX, M. RENÉ, M. J. BENISTON, S. A. BROWN, E. H. S. BURHOP, D. H. DAVIS, D. FERREIRA, E. FROTA-PESSOA, W. B. LASICH, N. N. RAINA, M. C. AMERIGHI, A. BONETTI, M. DI CORATO, C. C. DILWORTH, C. A. FEDRIGHINI, E. QUERCIGH, A. E. SICHIROLLO and G. VEGNI: *Nuovo Cimento*, **19**, 1077 (1961).

(*) This sample has been taken from that used in the previous work (¹), and the scanning procedure is described there. All particle tracks having a blob density at the K star ⩾ 30 blobs/100 μm and with dip ⩽ 40⁰ were followed to their end points.

which a fast particle is emitted unaccompanied by a charged π-meson or a slow Σ-hyperon of kinetic energy less than 60 MeV. Stars in which a hyperfragment was present were also excluded. The mass spectrum shown in Fig. 1 is obtained by measuring ionization on 432 suitable tracks with dip angle <30° and range ⩾ 1 cm, protons, deuterons and tritons with energies ex-
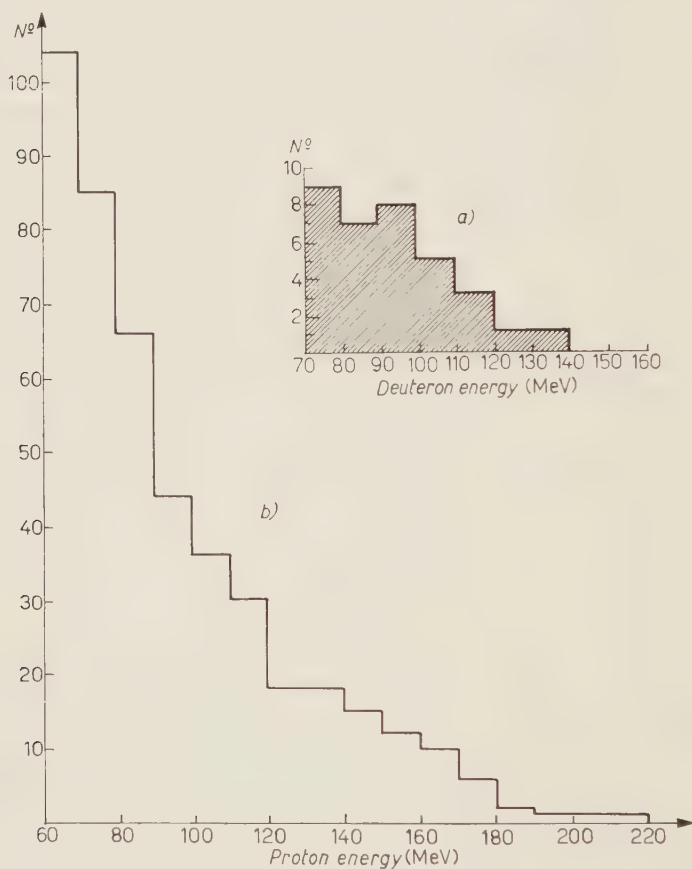

Fig. 2. – Energy distributions observed for the emission from the K⁻-interactions of (a) deuterons and (b) protons.

ceeding 52 MeV, 71 MeV and 84 MeV respectively included. Thirty-four particles with measured masses between 1.5 and 2.5 proton masses, were classified as deuterons, and five other for which the measured masses lay between 2.5 and 3.5 proton masses, as tritons. The energy spectrum for the deuterons is shown in Fig. 2a. For comparison the spectrum for high energy protons is shown in Fig. 2b. The energies of the observed tritons were 90, 100, 102, 113 and 123 MeV respectively.

Table I gives a summary of the data analysed and the relative abundance of p, d and t emissions as observed.

TABLE I. - *Relative abundance of* p, d *and* t *emitted with energy greater than* 84 MeV, *from* K⁻-*interactions at rest in emulsion* (*).

Total number of K⁻-interactions examined:	7 600
Total number of K stars in sample for which a baryon of angle of dip < 30° and range > 1 cm is emitted without accompanying π meson, hyperfragment or slow Σ hyperon (< 60 MeV)	432
Number of protons (**) of energy > 84 MeV in sample	180
Number of deuterons of energy > 84 MeV in sample	23
Number of tritons of energy > 84 MeV in sample	5

(*) A triton of energy 84 MeV has a range in emulsion of 1 cm.
(**) 76 particles leaving stack are not included.

3. – Search for the (Σ⁻n) hypernucleus.

From the above it is seen that energetic particles of mass two or three times that of the proton may be emitted from K⁻-meson absorptions at rest in complex nuclei. If the (Σ⁻n) hypernuclei can be formed it is reasonable to expect that they may be emitted with similar energies. The search for the (Σ⁻n) hypernucleus was conducted by examining fast Σ-hyperon tracks for evidence of anomalous mass values. 33 events were investigated in which a particle decayed in flight to a π-meson, and 3 events where a capture star was formed. In each case the energy of emission of the primary particle (estimated assuming the primary to be a Σ-hyperon) was not less than 60 MeV.

Two methods were used to estimate the mass:

a) *Ionization vs. range* – employed where a sufficient length of track was available (1 cm or more, 22 tracks out of 36).

b) *Scattering and ionization* – where the particle trajectory was suitably flat, *i.e.* dip angle < 20° and of length greater than 5 mm (26 tracks out of 36, of which 14 had length ≤ 1 cm. The other 12 tracks were measured by both methods, as indicated in Figs. 3 and 4).

No evidence for an anomalously high mass value was obtained from the sample. The results of the individual mass determinations by the two methods are shown in Figs. 3 and 4. Fig. 3 shows the masses obtained from a) (Ioni-
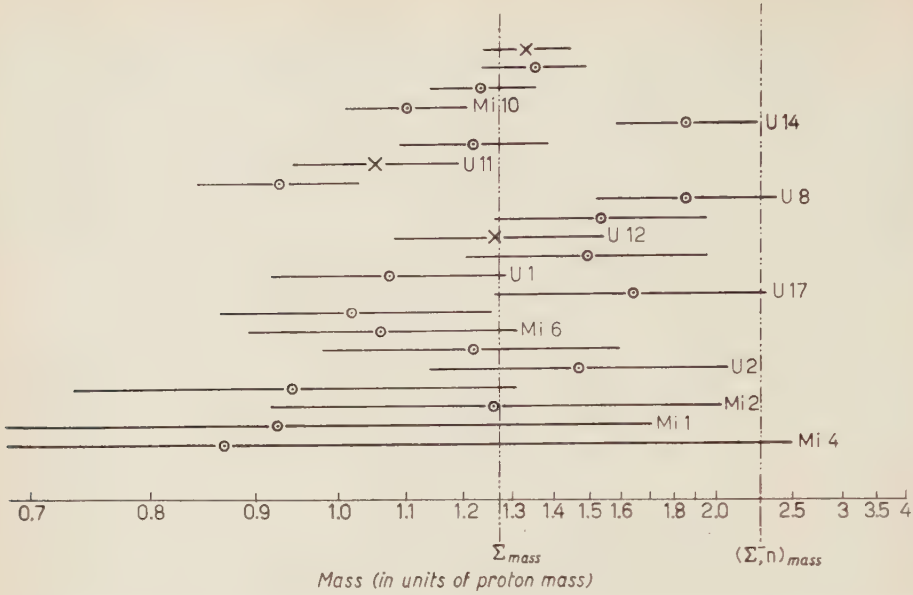
Fig. 3. – Mass estimates for hyperons as obtained from the ionisation-range characteristics of the tracks in the emulsion. The 12 events indicated by their index numbers were also measured by the scattering and ionization method (see Fig. 4). (⊙) Indicates decay in flight of hyperon to a π-meson. (×) Indicates interaction star formed by hyperon at rest.
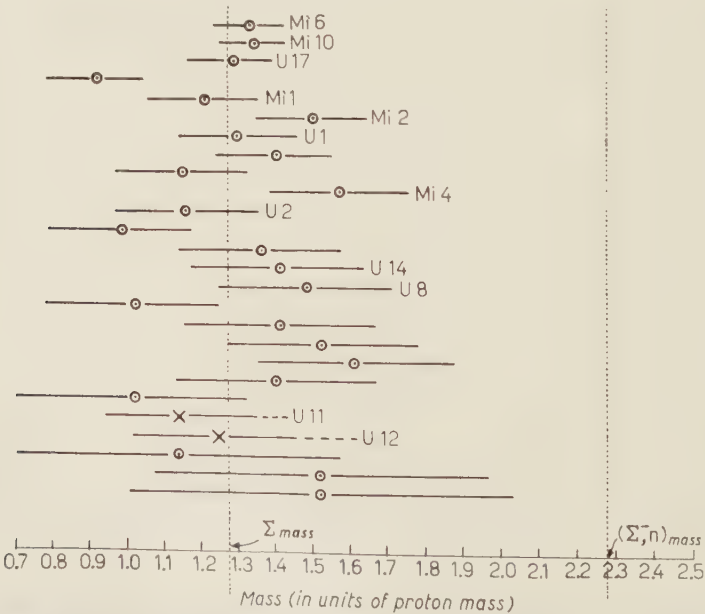

Fig. 4.   Mass estimates for hyperons as obtained from combined scattering and ionization measurements on tracks in emulsion. (⊙) Indicates decay in flight of hyperon to a π-meson. (×) Indicates interaction star formed by hyperon at rest.

zation-range) and Fig. 4 those from b) (scattering-ionization). Both mass
spectra are consistent with the assumption that all the particles have masses
of 1.3 proton masses, i.e. that of the Σ-hyperon. Thus no (Σ⁻n) hypernuclei
(2.3 times proton mass) can be discerned in our sample of 33 decays in flight
to π-meson (the proportion of Σ⁻-hyperons in this sample is estimated to
be about two-thirds or somewhat greater). The three fast hyperons producing
secondary stars after coming to rest in the emulsion were also found to have
the mass of the Σ-hyperon.

4. – Conclusions.

The study of the traces of fast particles from K⁻-meson interactions at
rest has shown that fast deuterons and tritons are emitted. The proportion
of deuterons among the stable particles (p, d and t) emitted with kinetic ener-
gies exceeding 84 MeV is $(11.1 \pm 2.5)\%$, whilst that of tritons is $(2.3 - 1.0)\%$ (*).
Fast deuterons and tritons of ranges greater than 1 cm are emitted from
$(1.0 \pm 0.2)\%$ of K⁻-meson interactions at rest in nuclear emulsion (**). Pos-
sibly the true yield is somewhat greater than this, since the fragments emitted
accompanied by a π-meson or slow Σ-hyperon would be excluded from the
present sample.

In the present study no (Σ⁻n) hypernucleus has been detected in a sample
of 36 fast Σ-hyperons. AMMAR et al. (²) have examined the primaries of
a) 15 decays in flight to π-meson and b) 84 events exhibiting the character-
istics of capture of hyperons. They have found all events of type a) and all
but four of type b) to be consistent with interpretation as Σ-hyperons. Al-
though the (Σ⁻n) interpretation could not be ruled out for 3 of the 4 events,
in no case was the identification certain (***). DAHL et al. (³) using a deute-

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(*) If it is assumed that the momentum distributions of protons, deuterons and
tritons are the same at emission the true proportion of deuterons and tritons would
be less than indicated here since a greater proportion of protons would be expected
to leave the stack.

(**) In a previous work (¹) it was suggested that the deuterons and tritons may
arise from the disintegration of α-particle clusters. However, their production may
also be accounted for by pick-up processes within the parent nuclei (see E. G. BELTRA-
METTI, L. CIUFFOLOTTI and G. TOMASINI: Nuovo Cimento, 11, 678 (1960)).

(²) R. G. AMMAR, N. CRAYTON, K. P. JAIN, R. LEVI-SETTI, J. E. MOTT, P. E.
SCHLEIN, O. SKJEGGESTAD and P. K. SRIVASTAVA: Phys. Rev., 120, 1914 (1960).

(***) It should be pointed out that the sample used in the work of AMMAR et al.
came mainly from one-nucleon captures of K⁻ mesons, while the sample described
in the present work refers to multinucleon captures. The non-observance of the (Σ⁻-n)
fragment in one of these samples does not necessarily esclude its observation in the
other.

rium bubble chamber have been unable in a sample of 227 events to detect any example of an event which was consistent with the dynamics of (Σ^-n) production, and conclude that if (Σ^-n) hypernuclei are produced in interactions of K^--mesons in deuterium the frequency is substantially less than 1 % (*). Thus the question of the existence of (Σ^-n) hypernuclei remains open. However, the investigations carried out so far indicate that if they do exist, then their occurrence must be relatively infrequent.

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(3) O. DAHL, N. HORWITZ, D. MILLER and J. MURRAY: *Phys. Rev. Lett.*, **4**, 428 (1960).

(*) One event, found in emulsion, was published as a possible example of Σ^-n decay (F. GANDOLFI, J. HEUGHEBAERT and E. QUERCIGH: *Nuovo Cimento*, **13**, 864 (1959)). The authors inform us that later measurements have shown that there were errors in the calibration of that event, and consequently the event should not be taken into consideration.

RIASSUNTO

In questo lavoro sono descritti i risultati di misure di massa sui barioni veloci emessi nella cattura a riposo di 7600 mesoni K^-. L'analisi è stata ristretta a quelle interazioni in cui non vengono emessi mesoni π carichi, nè iperoni Σ lenti (energia cinetica 60 MeV), nè iperframmenti. Come già riferito in un lavoro precedente (vedi nota (1)), un numero notevole di deutoni e tritoni è presente tra i barioni veloci: la proporzione dei deutoni di energia cinetica superiore a 84 MeV è di (11.1±2.5) %, quella dei tritoni è (2.3±1.0)%. I deutoni e tritoni veloci aventi percorsi maggiori di 1 cm vengono emessi da almeno (1±0.2)% delle interazioni di mesoni K^- a riposo nelle emulsioni nucleari. Tra i barioni veloci è stato misurato un certo numero di iperoni Σ di energia cinetica > 60 MeV: non è stato osservato alcun caso di ipernucleo (Σ n).

On the Gauge Invariant Theory
of Elementary Particle Interactions.

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Summary. — The weak interactions of elementary particles are derived from the principle of gauge invariance, following the general framework of Sakurai's investigations [3]. It is found that only one new boson, an isospinor of the second kind, has to be introduced. The actual weak processes can be obtained by combining the basic V-A interaction, which automatically gives an explanation for the $|\Delta t| = \frac{1}{2}$ and $|\Delta S| = 1$ rules, with various strong interactions. Next, a systematic study of the electromagnetic interaction is presented, indicating that, besides the photon field, the hypercharge — and the isospin — boson field also play a role, giving rise to short-range corrections for the familiar electromagnetic interaction. It is pointed out that the weak and electromagnetic interactions destroy the strict gauge invariance of the second kind for bosons which transmit the strong interactions, and therefore make it possible for them to have a non-vanishing rest masss. Finally, the consequences of hypercharge and isospin assignments for leptons are investigated, and it is shown that such an assignment automatically leads to a « muon number » which is conserved in all interactions.

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1. – Introduction.

It has been realized in recent years that one of the most outstanding problems in the theory of elementary particles is to find the « *raison d'être* » of their various interactions. Pioneering work in this field has been performed by Yang and Mills [1], who pointed out that whenever a gauge-group of the

(1) C. N. Yang and R. L. Mills: *Phys. Rev.*, **96**, 191 (1954).

first kind exists, there must exist a vector field, coupled in a unique way to the particles which obey the gauge invariance, and that the original gauge-group must be enlarged to incorporate gauge transformations of the second kind. Much valuable work has then been done along these lines (^{2}), and recently SAKURAI (^{3}) came out with an elaborate theory of strong interactions based on gauge invariance arguments alone. He was able to deduce, at least in a semi-quantitative manner, a great number of properties of strong inter-actions. He did not, however, discuss in detail the weak interactions. Further-more, a stumbling block appears to lie in the way of the theory: one cannot see how the boson fields associated with the gauge-groups could acquire a rest-mass. It is the purpose of the present paper to elaborate on the gauge-theory of weak and electromagnetic interactions and thereby also to show a way out of the above mentioned difficulty.

First, we briefly recapitulate Sakurai's scheme.

Baryon number conservation is equivalent to invariance under the phase-transformation

$$(1.1a) \qquad\qquad \psi \to \exp[ib\beta]\psi\,,$$

where b is the baryon-charge and β a phase. The concept of a local field forces us to consider $\beta = \beta(x)$ to be an arbitrary function of space-time co-ordinates. But then, to ensure invariance under this *gauge*-transformation, we must add to the free Lagrangian the interaction term

$$(1.1b) \qquad\qquad L'_B = ibJ^B_\mu B_\mu\,,$$

where J^B is the baryon current and B_μ is a Hermitian vector field subject to the gauge transformation of second kind,

$$(1.1c) \qquad\qquad B_\mu \to B_\mu + \partial_\mu\beta\,.$$

In a similar manner, hypercharge conservation is the result of invariance under

$$(1.2a) \qquad\qquad \psi \to \exp[iy\eta]\psi\,,$$

where y is the hypercharge. If η is an arbitrary function of x, we then must

(^{2}) See, for example: R. UTIYAMA: *Phys. Rev.*, **101**, 1597 (1956); T. D. LEE and C. N. YANG: *Phys. Rev.*, **98**, 1501 (1955); Y. FUJII: *Progr. Theor. Phys.*, **21**, 232 (1959); V. GUPTA: *Nuovo Cimento*, **19**, 586 (1961); A. SALAM and J. C. WARD: *Nuovo Cimento*, **11**, 568 (1959); **19**, 165 (1961).

(^{3}) J. J. SAKURAI: *Ann. of Phys.*, **11**, 1 (1960).

have the interaction

(1.2b)
$$L'_Y = iy J^Y_\mu Y_\mu ,$$

where J^Y_μ is the hyper-current and Y_μ is subject to the compensating transformation

(1.2c)
$$Y_\mu \to Y_\mu + \partial_\mu \eta .$$

Finally, isospin conservation follows from invariance under

(1.3a)
$$\psi \to \exp [it\boldsymbol{\theta\vartheta}]\psi ,$$

where t is the « iso-charge » and $\boldsymbol{\theta}$ denotes teh isospin operator (*i.e.*, $\boldsymbol{\theta} = \frac{1}{2}\boldsymbol{\tau}$ for isospinors, where $\boldsymbol{\tau}$ is the Pauli spin matrix, $\boldsymbol{\theta} = \boldsymbol{\xi}$ for isovectors, where $\boldsymbol{\xi}$ is the 3×3 Kemmer spin-one matrix). Considering the phase $\boldsymbol{\vartheta}$ as a space-time function, we are lead to the interaction

(1.3b)
$$L'_T = it \boldsymbol{J}^T_\mu \boldsymbol{T}_\mu ,$$

where $\boldsymbol{J}'_T$ is the iso-current and $\boldsymbol{T}_\mu$ a three-component (isovector) boson field, subject to the transformation

(1.3c)
$$\boldsymbol{T}_\mu \to \boldsymbol{T}_\mu + \partial_\mu \boldsymbol{\vartheta} .$$

The interactions (1.1b), (1.2b) and (1.3b) are considered as the *only* basic strong interactions; all customary strong interactions between various particles are merely phenomenological consequences (valid only in a limited energy range) of the above basic interactions.

SAKURAI estimated that the squared values of all three coupling constants b, y, t (which now also play the role of « charges ») lie in the range 1 to 20 (units: $\hbar = c = 1$), and that $b^2 > y^2 > t^2$.

Because of the respective gauge invariances of the second kind, the boson fields B and Y cannot have a restmass (*). Since, as is well known, this circumstance would entail catastrophic consequences, we must find some mechanism which effectively breaks down the strict gauge invariance (1.1c), (1.2c). We shall see that the systematic treatment of weak and electromagnetic interactions within the gauge-framework automatically serves this purpose.

(*) This statement does not necessarily apply to the $\boldsymbol{T}$ field, which has direct self-interactions; see (1).

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2. – Weak interactions.

We shall deduce the form of the basic weak interaction between fermions along the same lines as has been done for the strong interactions.

SAKURAI ([3]) pointed out that the « true fermion state » projection operator is $(1+\gamma_5)$. Hence, in the absence of other interactions, fermion number conservation is the consequence of invariance under the transformation

$$(2.1) \qquad \begin{cases} \psi = \exp\left[if(1+\gamma_5)\varphi\right]\psi\,, \\ \overline{\psi} \to \overline{\psi}\exp\left[-if(1-\gamma_5)\varphi\right]\,, \end{cases}$$

where f is the universal « fermion charge ». If q is a constant then this leads to the conservation of the fermion current

$$(2.2) \qquad J_\mu^F = f\sum \overline{\psi}\gamma_\mu(1+\gamma_5)\psi\,,$$

where the summation is extended over all fermion fields. If, however, in the spirit of local field theory, we consider q to be an arbitrary space-time function, then the free fermion Lagrangian will go over under (2.1) into

$$(2.3) \qquad L_0 \to L_0 - if\overline{\psi}\gamma_\mu\psi\partial_\mu\varphi - if\overline{\psi}\gamma_\mu\gamma_5\psi\partial_\mu\varphi\,.$$

We wish to compensate for these additional terms via the introduction of an interaction term. Since we assume that all bosons have definite parity, we shall need two separate bosons to mediate the interaction.

We first consider the vector part of the interaction. It was first suggested by GELL-MANN and FEYNMAN ([4]) that at least for non-strange particles this vector part exhibits a particular symmetry; it contains a term which conserves the first component of isospin, and another term which conserves its second component. (Actually, the conserved-current hypothesis is somewhat more general, but we shall use only the above restricted invariance-demand.) We want to account for this symmetry property by the boson which will be related to the vector part of the weak interaction. Since only two components of isospin have to be conserved, the corresponding boson must have two components, *i.e.*, it must be an isospinor.

We now proceed as follows. We demand for all fermions which are isospinors of the first kind, the invariance under

$$(2.4a) \qquad \psi \to \exp\left[if\tau_1\omega_1\right]\psi\,.$$

([4]) M. GELL-MANN and R. FEYNMAN: *Phys. Rev.*, **109**, 193 (1958).

If ω_1 is a constant then this leads to the conservation of the current $\bar{\psi}\gamma_\mu\tau_1\psi$. But if $\omega_1 = \omega_1(x)$ then

$$(2.5a) \qquad\qquad L_0 \to L_0 - if\bar{\psi}\gamma_\mu\tau_1\psi\partial_\mu\omega_1 \,.$$

To compensate for the extra term, we introduce the Hermitian interaction

$$(2.6a) \qquad\qquad L'_{V_1} = i\,\frac{f}{2}\,\bar{\psi}\gamma_\mu(\vec{\tau}_1 W_\mu + W^\dagger_\mu\overleftarrow{\tau}_1)\bar{\psi} \,,$$

where W_μ is the (non-hermitian) isospinor bosonfield mentioned above. If we now amend (2.4) by putting simultaneously

$$(2.7a) \qquad \begin{cases} \vec{\tau}_1 W_\mu \to \vec{\tau}_1 W_\mu + \partial_\mu\omega\vec{\tau}_1 \,, \\[2mm] W^\dagger_\mu\overleftarrow{\tau}_1 \to W^\dagger_\mu\overleftarrow{\tau}_1 + \overleftarrow{\tau}_1\partial_\mu\tilde{\omega} \,, \end{cases}$$

where

$$\omega = \begin{pmatrix} \omega_1 \\ 0 \end{pmatrix},$$

then, obviously, $L_0 + L'_{V_1}$ remains invariant.

In a similar manner, to ensure invariance under

$$(2.4b) \qquad\qquad \psi \to \exp[if\tau_2\omega_2]\psi$$

for arbitrary $\omega_2 = \omega_2(x)$, we have to introduce the interaction

$$(2.6b) \qquad\qquad L'_{V_2} = i\,\frac{f}{2}\,\bar{\psi}\gamma_\mu(-i\vec{\tau}_2 W_\mu + iW^\dagger_\mu\overleftarrow{\tau}_2)\psi \,,$$

and then subject W_μ to the gauge transformation of second kind

$$(2.7b) \qquad \begin{cases} \vec{\tau}_2 W_\mu \to \vec{\tau}_2 W_\mu + i\partial_\mu\omega\vec{\tau}_2 \,, \\[2mm] W^\dagger_\mu\overleftarrow{\tau}_2 \to W^\dagger_\mu\overleftarrow{\tau}_2 - i\overleftarrow{\tau}_2\partial_\mu\tilde{\omega} \,, \end{cases}$$

with

$$\omega = \begin{pmatrix} 0 \\ \omega_2 \end{pmatrix}.$$

(The i-factors in (2.6b) have been introduced for the sake of convenience.)

If we now add (2.6a) to (2.6b) and recall that $\tau_1 \pm i\tau_2 = 2\tau_\pm$, we obtain

for the vector part of the weak interaction

$$(2.8) \qquad L'_V = if\overline{\psi}\gamma_\mu(\tilde{\tau}_- W_\mu + W_\mu^\dagger \tilde{\tau}_+)\psi \, .$$

We emphasize that, in isospace, this is not a scalar but an *isospinor*. Hence it automatically will give rise to $|\Delta t| = \frac{1}{2}$ *transitions*. An isospinor weak interaction Lagrangian has indeed been often proposed (*), but in our case it directly follows from gauge invariance arguments.

Further information about the isobaric properties of (2.8) will be obtained if we can more closely specify the nature of the W-field. To this end we notice that, because of electric charge conservation, only the neutral component of W_μ should appear in (2.8). This is guaranteed if W_μ is an *isospinor of the second kind*, with negative hypercharge. Then, because of the well-known relationship $q = t_3 + \frac{1}{2}u$, the $t_3 = +\frac{1}{2}$ member is uncharged, while the $t_3 = -\frac{1}{2}$ member is negative. Hence

$$W_\mu - \begin{pmatrix} W_\mu^0 \\ W_\mu^- \end{pmatrix}, \qquad W_\mu^* \quad \overline{W}_\mu \quad \begin{pmatrix} \overline{W}_\mu^0 \\ W_\mu^- \end{pmatrix},$$

and, in (2.8) τ_- projects W_μ^0 from W_μ and likewise τ_- projects $\overline{W}_\mu^0$ from $W_\mu^\dagger$. The fact that for the W-field $u = -1$, will be of importance later on. We now stress only that (2.8) *automatically gives the* $|\Delta S| = 1$ *rule*.

Because L'_V is the sum of two conserved currents for the case of isospinor fermions of the first kind, it follows that the vector part of the weak interactions *of the non-strange particles* will not be renormalized. Since, however, we accept (2.8) as generally valid, and since for the strange particles it is obviously not related to conserved currents, it follows that the weak interactions of the strange particles' renormalization effects may play an important role. This particular feature of (2.8) may explain the observed scarcity of leptonic strange particle decays.

We should mention here that, although in the weak interactions only the neutral component of the W-boson takes part, obviously both components will have strong interactions, mediated through the Y and $\boldsymbol{T}$ fields, to which W is coupled by its y and t charge.

We now turn to the axial vector part of the weak interaction. This part does not exhibit any new type of symmetry in itself; hence, as opposed to all previous cases, we do not have to introduce a new boson. On the other hand, it must be a neutral particle. Further, in order to couple the basic weak interaction to as many particles as possible (see below), it must be a boson which can interact in many ways with equal strength. Hence, the choice is fairly

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(*) See, for example, B. D'ESPAGNAT: *Nuovo Cimento*, **18**, 287 (1960).

obvious; we submit that the axial vector part of the weak interaction is related to the B-field, and put

$$(2.9) \qquad\qquad L'_A = if\overline{\psi}\gamma_\mu\gamma_5 B_\mu\psi \ .$$

Adding this to (2.8), we obtain the complete basic weak interaction

$$(2.10) \qquad L'_w = if\overline{\psi}\gamma_\mu(\vec{\tau}_- W_\mu + W^\dagger_\mu\vec{\tau}_+)\psi + if\overline{\psi}\gamma_\mu\gamma_5 B_\mu\psi \ .$$

We must now see how this interaction compensates for the change (2.3) introduced in the free fermion Lagrangian by the fermion gauge transformation (2.1). We find that the complete Lagrangian, $L = L_0 + L'_w$, is invariant if we amend (2.1) by the simultaneous gauge transformationss

$$(2.11) \qquad \begin{cases} B_\mu \rightarrow B_\mu + \partial_\mu\varphi \ , \\[4pt] W_\mu \rightarrow W_\mu + \vec{\tau}_+\partial_\mu\varphi \ , \\[4pt] W^\dagger_\mu \rightarrow W^\dagger_\mu + \partial_\mu\widetilde{\varphi}\vec{\tau}_- \ , \end{cases}$$

where, in the last two lines,

$$\varphi \equiv \frac{1}{2}\begin{pmatrix} \varphi \\ \varphi \end{pmatrix} \ .$$

Hence, we have succeeded in deducing the form of the weak interaction Lagrangian from gauge invariance arguments.

It now remains to be seen, how (2.10) can account for the various actual weak processes between the familiar « charged-uncharged » currents in the Puppi–Gell-Mann diagram. It is obvious that the actual processes will come about by combining the B or W fields emitted in the basic weak interaction with strongly interacting vertices. Fig. 1 shows, as a typical example, the
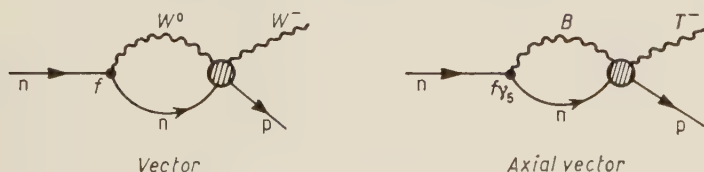

Vector   Axial vector

Fig. 1.

vector and axial-vector part of the ($\overline{p}n$) current; while Fig. 2 illustrates the corresponding ($\overline{p}\Lambda$) currents. Combining such « currents with a prong » we can account for the actual processes, keeping in mind also that interference

effects between the V and A currents will play an essential role. To present details, of course, is just as hopeless, at the present stage, as it is for the strong interactions in Sakurai's work.
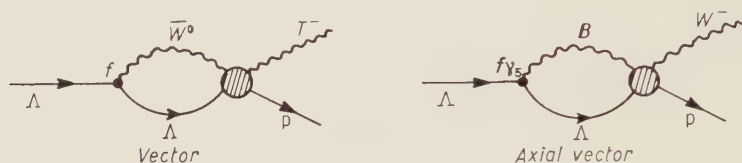

Fig. 2.

Leptonic processes can easily be incorporated into the scheme if we assign isospin and hypercharge to leptons (*). We then obviously must join the electron and the neutrino in a doublet, and a similar doublet must exist for the muon. Since a particle can not appear in two different multiplets, we are naturally led to the existence of two different neutrinos, ν_e and ν_μ, as has been often suggested recently. The V- and A-parts of the $(\overline{e^-}\nu_e)$ current are shown in Fig. 3, and similar graphs hold for the $(\overline{\mu^-}\nu_\mu)$ current.
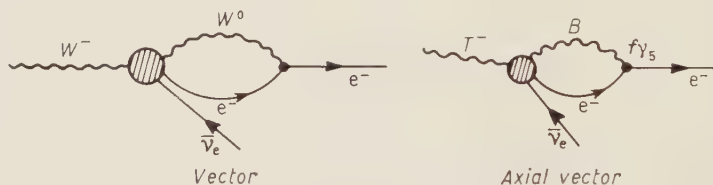

Fig. 3.

We now come to an important point. Let us consider the complete Lagrangian of baryons

$$(2.12) \qquad L = L_0 + ib\overline{\psi}\gamma_\mu B_\mu\psi + if\overline{\psi}\gamma_\mu\gamma_5 B_\mu\psi + \dots,$$

where the dots indicate all other interaction terms which do not contain the B-field. We see that (2.12) *is no longer invariant under the baryon gauge-group*; the presence of the third weak interaction term breaks down the invariance and (1.1c) no longer holds. Hence, in the presence of weak interactions the B-field does not admit a gauge transformation of the second kind, and *there is no longer any reason why the B-field cannot acquire a mass.* Similarly, the

(*) We shall comment on some implications of such an assignment in the concluding section of this paper.

fermion-gauge transformation of B is broken down by the strong interaction. Of course, both baryon number and fermion number conservation still hold true because (2.12) is still invariant under either the baryon *phase*-transformation or the fermion *phase*-transformation group (with β and φ constant).

It may be asked why the baryonic gauge invariance cannot be maintained in the presence of weak interactions. A tentative answer may be furnished by the following argument. The augmentation of the barionic *phase*-group to a *gauge*-group was motivated by the observation that relative phases of baryons cannot be measured, whatever their space-time separations. However, the weak interactions lead to decay processes, and although they conserve baryon number, they in some manner provide a means of determining the baryon phase. Or, in another formulation: the weak interactions, by implying decay processes, prevent a perfect localization in space-time, hence the argument about the local character of the fields cannot be maintained.

At this point we mention that the W-field will also have a mass, since in addition to its role in the weak interaction, it is also coupled to the Y and T fields. Hence, the total Lagrangian $L_0 + L_V + iy J_\mu^{W(Y)} Y_\mu + it J_\mu^{W(T)} T_\mu$ does not permit a gauge invariance of the second kind for the W-field.

We conclude this section with an estimate of the (unrenormalized) coupling constant f. Since the actual weak interactions come about by combining two elementary interactions (2.10), the effective current-current interaction will have the proportionality factor $f^2/\varkappa^2$, where $\varkappa$ is the mass of the mediating boson. Equating this with the observed weak coupling constant $g \simeq 10^{-49}$ erg cm^3 and taking $\varkappa \sim 5\varkappa_\pi$, we obtain, in units $\hbar = c = 1$,

$$f^2 \approx 10^{-5}.$$

Because of the role played by the strong interactions in obtaining the actual charged-uncharged currents, f^2 may actually be one order of magnitude smaller than this value.

3. – Electromagnetic interactions.

Electric charge conservation means the conservation of the current (*)

$$J_\mu = e \sum \overline{\psi} \gamma_\mu \psi ,$$

where the summation is extended over all such *members* of the various multiplets that have an electric charge. This conservation law is accounted for

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(*) For brevity we restrict ourselves to fermions.

if we demand invariance under

(3.1) $$\psi \to \exp\left[ie\varepsilon\right]\psi$$

for the charged *members* of all possible multiplets. This situation appears to be unaesthetic, because all other gauge transformations refer to *complete* multiplets. True enough, one can replace (3.1) by

(3.2) $$\psi \to \exp\left[ieQ\varepsilon\right]\psi$$

which refers to complete multiplets, and where

(3.3) $$Q = \theta_3 + \tfrac{1}{2}U$$

is the charge operator of the multiplet, which projects the charged members. But then we have to face a somewhat mysterious situation. Charge conservation should follow from gauge invariance alone, and we cannot believe that charge is conserved because the sum of t_3 and $\tfrac{1}{2}u$ is conserved. Besides, the electromagnetic interactions conserve t_3 and u *separately*. It has been suggested that charge conservation is a consequence of reflection invariance on the $X_1 X_2$ plane of isospace. However, this gives charge conservation up to modulo 2 only; and also, we believe that any conservation law is the sole consequence of a gauge group. Hence we are motivated to change the argument.

We first remark that we expect that in electromagnetic interactions t_3, u *and* q are conserved separately. Next we notice that the presence of θ_3 and U in the charge projector hints toward some role of the T^3 and Y fields in the electromagnetic interaction; otherwise it would be inconceivable for these operators to emerge in the gauge transformation.

With these ideas in mind, we now demand invariance under

(3.4) $$\psi \to \exp\left[ie(\theta_3 + \tfrac{1}{2}U + Q)\varepsilon\right]\psi$$

for every multiplet. Here θ_3 is $\tfrac{1}{2}\tau_3$ or ξ_3, U is the 2×2 unit matrix or minus one times this matrix for isospinors of the first and second kind respectively, and Q is

(3.5) $$\begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{for isospinors of the first kind}, \\[2ex] \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} & \text{for isospinors of the second kind}, \\[2ex] \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & \text{for isovectors}. \end{cases}$$

Thus we retain the geometrical meaning of Q as the generator of a reflection operator, but do not demand its conservation because of invariance under this operation.

If the phase ε is a constant then (3.4) gives rise to the existence of the conserved current

$$(3.6) \qquad J_\mu^e = J_\mu^{T_3} + \tfrac{1}{2}J_\mu^Y + J_\mu^Q\,,$$

where J_μ^Y is the already known hyper-current and $J_\mu^{T_3}$ the third component of the $\boldsymbol{J}_\mu^T$ isocurrent, while J_μ^Q is a new current,

$$(3.7) \qquad J_\mu^Q = \sum \bar\psi \gamma_\mu Q \psi\,,$$

with Q given by (3.5).

Now, if we enlarge the phase transformation to become a gauge transformation, $i.e.$, if ε is an arbitrary function; then the free Lagrangian changes according to

$$(3.8) \qquad L_0 \to L_0 - ie\,J_\mu^e\,\partial_\mu\varepsilon \equiv L_0 - ie\,J_\mu^{T_3}\partial_\mu\varepsilon - i\,\frac{e}{2}\,J_\mu^Y\partial_\mu\varepsilon - ieJ_\mu^Q\partial_\mu\varepsilon\,.$$

Since the continued conservation of J^{T_3} and J^Y is already guaranteed by the existence of the T^3 and Y fields, we have to introduce only one more boson field, A_μ, and put

$$(3.9) \qquad L_e' = ieJ_\mu^{T_3}T_\mu^3 + i\,\frac{e}{2}\,J_\mu^Y Y_\mu + ieJ_\mu^Q A_\mu\,,$$

subjecting the three boson fields to the simultaneous gauge transformations of the second kind,

$$(3.10) \qquad \begin{cases} J_\mu^{T_3} \to J_\mu^{T_3} + \partial_\mu\varepsilon\,, \\[4pt] Y_\mu \to Y_\mu + \partial_\mu\varepsilon\,, \\[4pt] A_\mu \to A_\mu + \partial_\mu\varepsilon\,. \end{cases}$$

Then the total Lagrangian $L = L_0 + L'$ is indeed invariant under (3.4) and (3.10).

We notice an analogy between the weak and electromagnetic interaction: in contrast to the various basic strong interactions, they contain *several* boson fields. This, in a sense, makes up for the fact that, while for the weak and electromagnetic interactions we need *one* gauge invariance for each, « *the* » strong interaction demands three different gauge invariances.

When we now consider the *complete* Lagrangian of all charged matter, *i.e.*

$$(3.10) \qquad L = L_0 + L'_e + L'_Y + L'_{T_3} = L_0 + ieJ^Q_\mu A_\mu + i\frac{e}{2}J^Y_\mu Y_\mu +$$
$$+ ieJ^{T_3}_\mu T^3_\mu + iyJ^Y_\mu Y_\mu + itJ^{T_3}_\mu T^3_\mu + \dots ,$$

we immediately see that the presence of the electromagnetic terms breaks down the Y-field gauge invariance of the second kind. The same happens with the T^3-gauge invariance. Hence, because of the presence of the electromagnetic interaction (3.9), *there is no longer a reason why the Y (and T_3) field could not acquire a restmass*. Thus, the above systematic treatment of the electromagnetic interaction also removes this stumbling block.

(3.11) is not invariant under the electromagnetic gauge group either. Hypercharge, t_3, and electric charge conservation, of course, still hold true because (3.11) is invariant under the respective *phase*-transformations.

We further notice that (3.11) still permits a « photon gauge group »

$$(3.12) \qquad \begin{cases} \psi \ \rightarrow \exp\left[ieQ\varepsilon\right]\psi \,, \\ A_\mu \rightarrow A_\mu + \partial_\mu\varepsilon \,. \end{cases}$$

Hence, the *photon field A_μ is strictly massless*.

An important consequence of our electromagnetic interaction (3.9) is that the conventional electromagnetic forces (mediated by A_μ) will have a short-range correction (due to the Y_μ and T^3_μ term). Because of the presumably large Y- and T-mass ($\varkappa_r \sim \varkappa_T \approx 5\varkappa_\pi$, according to Sakurai's estimate from various strong phenomena), these corrections would show up only at distances below 10^{14} cm. Presumably, interference effects between the various terms of the electromagnetic interaction would play a role and quantum electrodynamics at small distances could be drastically modified. A particular consequence of these short range electromagnetic forces is that even the neutral member ($q = 0$, $t_3 \neq 0$, $u \neq 0$) of isospinor doublets will experience short-range electromagnetic effects. This could have an important part in explaining the electromagnetic structure of such particles (neutron, K^0-meson).

4. – Further discussion and comments.

We have seen that a systematic application of the theory of conservation laws and associated gauge groups enables us to derive explicit and consistent forms of the weak and electromagnetic interaction. At the same time these interactions have the effect of breaking down the absolute validity of the gauge

invariance of the second kind for the bosons that transmit the interactions and hence make it possible for them to acquire a non-vanishing restmass. From the « moralistic » point of view, it seems to be gratifying to observe that the relative weakness of the weak and electromagnetic interactions is not a derogatory epithet: although they are weak, it is their very existence that makes the whole gauge-scheme of interactions consistent!

An interesting feature of our scheme is that all bosons, with the exception of the photon A_μ, participate in both « strong » and « weak » interactions. They share this property with the fermions. This fact may be important if we believe in a composite particle model, where the basic building blocks are fermions.

To make the whole scheme work we must assign hyper-charge and iso-charge to the leptons. Although there are quite a number of previous known attempts to assign isobaric properties to leptons, in the gauge invariant theory of interactions such an assignment will have drastic consequences. This is so because now the isospin and hypercharge also play the role of coupling constants, hence the leptons will participate in the iso-current J_μ^τ and hyper-current Y_μ and may interact rather strongly with baryons. It must be checked as to whether such interactions will not lead to observable, drastic corrections. To get a vague idea about the order of magnitude of these y- and t-interactions of leptons, we assume that, for a reasonable range of energies, the short-range forces due to these interactions can be described by static, Yukawa type potentials, and that we can calculate cross-sections in the Born approximation.

As is well known from elementary scattering theory, the differential cross-section for an electron with incident energy E, scattered on a nucleon at rest due to the static Yukawa type potential

$$V = - y^2 \frac{\exp\left[-\varkappa_y r\right]}{r},$$

will be

(4.1)
$$\sigma_y(\vartheta) = \frac{m^2 y^4 \hbar^2 c^2}{16 m^2 E^2 (\hbar^2 \varkappa_y^2 / 4mE + \sin^2 \vartheta/2)^2},$$

and the total cross-section is

(4.2)
$$\sigma_y = \frac{m^2 y^4 \hbar^2 c^2}{2\hbar^4 \varkappa_y^4 (1 + 4Em/\hbar^2 \varkappa_y^2)},$$

where m is the electron mass, $\varkappa_y = m_y c/\hbar$, and y the (dimensionless) coupling constant to the Y-field. Now, since $\varkappa_y \approx 5\varkappa_\pi$, we can neglect $4Em/\hbar^2 \varkappa_y^2$ relative to 1, up to energies of about 10^6 MeV. Hence, the cross-section will be

largely independent of energy. Then, using $y^2 \approx 4$, (which is certainly not an underestimate), we obtain

$$(4.3) \qquad\qquad \sigma_y \approx 10^{-33} \text{ cm}^2 .$$

This is certainly too small to be readily observed. If the scattered particle is a muon, σ_y will be about ten thousand times bigger, but still unobservable. The effect of the t-coupling caused scattering will be even smaller by an order of magnitude, and interference effects will also arise.

Much more interesting and crucial is the comparison of the y (and t) scattering with ordinary Coulomb scattering. Since the Coulomb scattering cross-section is

$$(4.4) \qquad\qquad \sigma_{\text{Coul}}(\vartheta) = \frac{e^4 \hbar^2 c^2}{4E \sin^4 \vartheta/2} ,$$

the ratio of y to Coulomb scattering of an electron on a proton becomes, due to (4.1) and (4.4)

$$(4.5) \qquad\qquad \sigma_y(\vartheta)/\sigma_{\text{Coul}}(\vartheta) = \frac{y^4}{4e^4} \frac{1}{E} \frac{\sin^4 \vartheta/2}{(\hbar^2 \varkappa_y^2/4mE + \sin^2 \vartheta/2)} .$$

Here again, up to energies of 10^6 MeV, we can neglect $\sin^2 \vartheta/2$ with respect to $\hbar^2 \varkappa_y^2/4mE$, and then, measuring the energy E in MeV, we obtain

$$(4.6) \qquad\qquad \sigma_y(\vartheta)/\sigma_{\text{Coul}}(\vartheta) \approx 10^{-2} E_{\text{MeV}} \sin^4 \vartheta/2 .$$

At energies up to 10^3 MeV and angles $\vartheta < 20°$, this ratio is almost completely negligible. However, for very large angle scattering the ratio approaches and even exceeds unity. For the case of muons the ratio is even much greater. Although all these calculations are extremely rude, they indicate that at moderate energies we need not have fear of measurable effects. On the other hand, it seems plausible to account for the oft-alleged discrepancy in high energy—large angle μ-p scattering from the Coulomb law by means of the y-interaction.

Returning now to the isospin and hypercharge assignment to leptons, we recall from Section 2 that, in order to account for the leptonic weak interactions, we must have an (e^-, ν_e) and a (μ^-, ν_μ) doublet. Since both are then isospinors, the only way to distinguish between them is to assign different signs of hypercharge. But then, since e^- and μ^- (and *not* μ^+) are « particles », the customary charge displacement formula $q = t_3 - \frac{1}{2}u$ ceases to work. This fact, however, allows for an interesting speculation. We propose that

$$(4.7) \qquad\qquad q = t_3 + \tfrac{1}{2}u + n$$

and take the (e^-, ν_e) as an isospinor of the second kind $(u = -1)$ and $n_e = 0$, while we consider (μ^-, ν_μ) as an isospinor of the first kind and put $n_\mu = -1$. We then have the following assignments:

	t_3	$u/2$	n	q
ν_e	$+\tfrac{1}{2}$	$-\tfrac{1}{2}$	0	0
e^-	$-\tfrac{1}{2}$	$-\tfrac{1}{2}$	0	-1
ν_μ	$+\tfrac{1}{2}$	$+\tfrac{1}{2}$	-1	0
μ^-	$-\tfrac{1}{2}$	$+\tfrac{1}{2}$	-1	-1

The new quantum number n can now be identified with the « muon number » first introduced by SCHWINGER [5]. This quantum number is strictly conserved because the total charge q is conserved, as is $t_3 + \tfrac{1}{2}u$. The conservation of the latter for strong and electromagnetic interactions is evident, and it also holds true for the weak interactions, because, as we have seen, only W^0_μ takes part in them, and for this particle $t_3 = +\tfrac{1}{2}$, $u/2 = -\tfrac{1}{2}$, hence $\Delta(t_3 + \tfrac{1}{2}u) = 0$. Thus, we are automatically led to the *conservation of muon number*. It then follows, for example, that we must have

$$(4.8) \qquad\qquad \mu^- \to e^- + \bar{\nu}_e + \nu_\mu \,,$$

and, indeed, using the arguments presented toward the end of Section 2, we can verify that the combination of our weak currents can lead only to process (4.8) and not to $\mu^- \to e^- + \nu_e + \bar{\nu}_\mu$, etc. Processes like $\mu \to 3e$, $\mu \to e + \gamma$, etc., are excluded primarily because of charge conservation and also because only W^0 participates in the basic weak interaction. We also notice that the appearance of the new quantum number n is a straightforward consequence of having disassociated charge conservation from its geometrical interpretation and of having insisted on i) gauge transformations and ii) meaningful isobaric assignments for the leptons. (The latter demand is unavoidable in order to incorporate leptonic weak interactions.)

A few words should be added about the masses of the leptons. Here, of course, we are on completely unsafe grounds. But one worry, *viz.*, that because of their y and t charge the leptons might acquire a large mass, can be easily appeased. It is known that in the present framework of field theory the self-energies are roughly proportional to the already present « bare » masses. Now, when we « assemble » a lepton from some fermionic « Urschmiere », in

[5] J. SCHWINGER: *Ann. of Phys.*, **2**, 407 (1957).

the sense of Sakurai's point of view, the bare mass will be very small, because the basic fermionic charge f is so many orders of magnitude smaller than the baryon charge b. Then, when adding the y and t charges, only relatively small self-energy effects will arise. But one thing is noteworthy—the mass difference between the electron and the muon can arise from the different sign of their y charge. Since the mass difference between N and Ξ must be due to the same reason, it seems interesting to note that the order of magnitude of these two mass differences is the same.

In concluding, we must pay attention to a possible danger point of our whole theory (*). Since the B field occurs both in strong and weak interactions, and in the latter case it has a pseudvector coupling, we might expect that by combining the weak L'_A and the strotg L'_B interactions, parity violating « strong » interactions could occur. To obtain a vague estimate about their relative importance, we compare, again in the first Born approximation, the scattering cross-sections between two baryons mediated i) by the B field, combining L'_B and L'_A (parity violating) and ii) by the pion field (parity conserving). Denoting these cross-sections by σ_1 and σ_2, respectively, we find that their ratio is largely independent of energy and obtain

$$\sigma_1/\sigma_2 \approx \frac{f^2 b^2}{G_\pi^4} \frac{\varkappa_\pi^4}{\varkappa_B^4}.$$

Using the values $b^2 \approx 20$, $G_\pi^2 \approx 15$, $\varkappa_B \sim 5\varkappa_\pi$ and $f^2 \sim 10^{-5}$ (see above), we have

$$\sigma_1/\sigma_2 \sim 10^{-9}.$$

Fortunately, this possible admixture of parity violation is certainly well below the observed limit. For phenomena involving K-mesons the ratio becomes a few hundred times larger.

A similar danger point arises from the circumstance that in our electromagnetic interaction (3.9) there is a term containing the T^3 field. Now, when this electromagnetic term combines with strong interactions, it could lead to isospin non-conserving corrections. Denoting the baryon-baryon scattering cross-section which ensues from combining L'_T with the $J_\mu^{T_3} T_\mu^3$ term in the electromagnetic interaction by σ_3, and comparing it to the pion-mediated interaction σ_2, we obtain for the ratio of isospin non-conserving to isospin conserving scattering

$$\sigma_3/\sigma_2 \approx \frac{t^2 e^2}{G_\pi^4} \frac{\varkappa_\pi^4}{\varkappa_T^4}.$$

(*) I am obliged to Prof. S. BARSHAY (Brandeis University) for having stressed to me the importance of this point.

Using Sakurai's estimate $t^2 \sim 1$, we then have

$$\sigma_3/\sigma_2 \sim 10^{-8} .$$

This ratio is again safely below the experimental limit. For K-meson pheno-
mena it becomes $\sim 10^{-6}$, and it is perhaps interesting to note that violation
of isospin conservation in K-phenomena has been reported from time to time.

Notes added in proof.

a) A relatively simple model in which b_0 corresponds to the sequence $\overline{\mathrm{K}} + \mathcal{N} \to$
$\to \pi'^0 + \Lambda \to Y_0^{**} \to \pi + \Sigma$ has been constructed which gives $n \leqslant 4$ (to be published).

b) An important contribution may also come from an incident $\overline{\mathrm{K}}$-$\mathcal{N}$ $P_{\frac{1}{2}}$-state going
to a strongly interacting state of two S-wave pions and a Λ.

RIASSUNTO (*)

Derivo le integrazioni deboli delle particelle elementari dal principio dell'invarianza
di gauge seguendo l'andamento generale dell'esame di Sakurai. Trovo che si deve intro-
durre un solo nuovo bosone, un isospinore di seconda specie. Gli effettivi processi deboli
si possono ottenere combinando l'interazione fondamentale $V-A$, che dà automatica-
mente una spiegazione delle regole $|\Delta t| = \frac{1}{2}$ e $|\Delta S| = 1$, con varie interazioni forti.
Poi presento uno studio sistematico della interazione elettromagnetica, indicando che,
oltre al campo fotonico, ha un suo ruolo anche il campo bosonico dell'ipercarica — e
dell'isospin —, che dà luogo a correzioni di breve range della nota interazione
elettromagnetica. Faccio notare che le interazioni debole ed elettromagnetica distrug-
gono la rigida invarianza di gauge di seconda specie per i bosoni che trasmettono le
interazioni forti, e quindi permettono che essi abbiano una massa di quiete che non
si annulla. Infine esamino le conseguenze delle assegnazioni di ipercarica ed isospin ai
leptoni, e mostro che tale assegnazione porta automaticamente a un « numero di muoni »,
che si conserva in tutte le interazioni.

Calculation of the Replacement Effect
in the Statistical Theory of Lattice Damage.

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Summary. — Calculations are made on the number of lattice damages due to irradiation by high energy particles. Using the assumption of [7] that the atoms responsible for the displacements due to collision may be captured in the lattice sites of the displaced atoms, the capture probability is written in the exponential form $\exp[-\alpha(E/E_d)]$ and the average number of lattice damages is obtained as $m_\lambda(E) \sim K(\alpha)((E/E_d)+1)$.

Introduction.

Lattice damages occur in solids due to irradiation with high-energy particles. Several authors [1-5] have given theoretical estimates for the average number of these lattice defects. The theoretical data are however, generally about five time greater than the averages deduced from actual experiments. In order to remove this discrepancy some authors have attempted to generalize the theory. Thus, for instance, PÁL and the author [7] have derived more general equations for the statistical theory of lattice damage by taking into account the replacement of the bombarding particle as well. The necessary

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[1] F. SEITZ and S. J. KOEHLER: *Solid State Physics*, Vol. 2 (New York, 1956), pp. 381.

[2] F. SEITZ and W. A. HARRISON: *Phys. Rev.*, **98**, 1530 (1955).

[3] W. S. SNYDER and J. NEUFELD: *Phys. Rev.*, **97**, 1636 (1955); **103**, 862 (1955).

[4] L. PÁL: *Energia és Atomtechnika*, **10**, 255 (1957).

[5] A. E. FEIN: *Phys. Rev.*, **109**, 1076 (1958).

[6] G. LEIBFRIED: *Nukleonik*, **1**, 57 (1958).

[7] L. I. PÁL and G. NÉMETH: *Nuovo Cimento*, **12**, 293 (1959).

computations have now been performed and are reported in this paper. It will be shown that the difference between theoretical estimates and experimental values can be considerably reduced if the replacement effect is taken into account.

1. – Computation of the average number of lattice defects.

As this paper is mainly the detailed elaboration of ideas presented in ([7]), the physical aspects of the problem will not be dealt with here and some symbols used in ([7]) will remain unchanged.

Without dealing with the process as a whole, we shall consider only the damage caused by the displaced atom. The expression for the average number of displaced atoms referred to in the following is to be found in any of the papers from ([1]) for ([7]). This expression gives the average number of atoms displaced by a particle having energy E as

$$(1) \qquad m_1(E) \sim \exp\left[-c\right]\left[\frac{E}{E_d}+1\right] \qquad c = 0.577 = \text{Euler constant}.$$

This is the formula to which our results will be compared.

Let us first recapitulate some of the results of ([7]) which would be of immediate use in the present problem.

Let $p_m^{(0)}(E)$ be the probability that with the possibility of replacement considered, the number of atoms displayed by a particle of energy E is m, The generating function $g_0(E;x) = \sum_{m=0}^{\infty} \exp\left[mx\right] p_m^{(0)}(E)$ $(x \leqslant 0)$ of $p_m^{(0)}(E)$ satisfies the eq. (7),

$$(2) \quad g_0(E;x) = \frac{1}{E}\int_0^E \left[1 - K(E-E')\right]g_0(E';x)\,\mathrm{d}E' +$$

$$+ \frac{1}{E}\int_0^E K(E-E')[(1-l(E'))g_0(E';x) + l(E')]g_0(E-E'-E_d;x)\,\mathrm{d}E',$$

where

$$K(E) = \begin{cases} 1 & E \geqslant E_d, \\ 0 & E < E_d, \end{cases}$$

and $l(E')$ is the probability that the atom responsible for the displacement (having now energy E') will be captured in the lattice site of the displaced atom.

In order to solve the eq. (2) let us introduce the Laplace transform

$$G(p\,;\,x) = \int\limits_0^{+\infty} \exp\left[-\,pE\right]g_0(E\,;\,x)\,\mathrm{d}E\ .$$

Multiplying now eq. (2) by $\exp\left[-\,pE\right]$ and integrating from 0 to ∞, we obtain

$$(3) \quad -\,G' = \frac{1 - \exp\left[-\,pE_d\right]}{p}\,G + \exp\left[-\,pE_d\right]G\ \cdot$$

$$\cdot\left\{G - \int\limits_0^{\infty} \exp\left[-\,pE'\right]l(E')\left[g_0(E'\,;\,x) - 1\right]\mathrm{d}E'\right\}.$$

The statements which can be made about the replacement probability $l(E')$ are only of a highly qualitative nature. Therefore let us write simply that

$$l(E') = 1 \qquad 0 < E' \leqslant E_d\ .$$

Thus, since $g_0(E\,;\,x) = e^x$ if $0 < E' \leqslant E_d$;

$$(4) \quad -\,G' = \frac{1 - \exp\left[-\,pE_d\right]}{p}\,G + \exp\left[-\,pE_d\right]G\left[G - (e^x - 1)\frac{1 - \exp\left[-\,pE_d\right]}{p}\right].$$

(4) is a Bernoullian differential equation which can be readily integrated and we shall write immediately the solution (considering the boundary conditions $\lim\limits_{p\,\to\,\infty} pG = e^x$),

$$(5) \qquad G(p\,;\,x) = E_d\psi(pE_d)\left\{\exp\left[-\,x\right] - \int\limits_{pE_d}^{\infty}\exp\left[-\,y\right]\psi(y)\mathrm{d}y\right\}^{-1},$$

where

$$\psi(y) = y^{-1}\exp\left\{-\,E_1(y) + (e^x - 1)\left[E_1(2y) - E_1(y)\right]\right\},$$

and

$$E_1(y) = -\,E_i(-\,y) = \int\limits_y^{\infty}\frac{\exp\left[-\,u\right]}{u}\,\mathrm{d}u\ ,$$

is the well known exponential integral.

From the expression (5) for $G(p; x)$ it is easy to determine the Laplace transforms of the moments by differentiating them with respect to x. After some reductions the average number of displacement is obtained as

$$(6) \qquad n_1(p) = \int_0^\infty \exp\left[-\,pE\right] m_1(E)\,\mathrm{d}E = \left.\frac{\partial G(p; x)}{\partial x}\right|_{x=0} =$$

$$= \frac{1 - \exp\left[-\,pE_d\right]}{p} + \frac{\exp\left[E_1(pE_d)\right]}{p} \int_{pE_d}^\infty \exp\left[-\,y - E_1(y)\right]\mathrm{d}y\,.$$

Hence, for $p \to 0$, we find

$$(7) \qquad n_1(p) = \exp\left[-\,c\right] K \left[\frac{1}{p^2 E_d} + \frac{1}{p}\right] + \ldots;\qquad K = \int_0^\infty \exp\left[-\,y - E_1(y)\right]\mathrm{d}y\,,$$

and upon retransformation we have

$$(8) \qquad m_1(E) \sim \exp\left[-\,c\right] K \left[\frac{E}{E_d} + 1\right],\quad c = 0.577 = \text{Euler constant (*)}\,.$$

By numerical integration the value of the constant K was found to be $K = 0.624$. Comparing this expression with (1), it is seen that, by taking into account the replacement, the average number of lattice defects has been reduced almost by a factor of 2.

Remark: Let us now take more generally $l(E') \neq 1$ for $0 \leqslant E' \leqslant E_d$, but with $l(E') = 0$ for $E' > E_d$. This leads by some easy computations to

$$m_1(E) \sim \exp\left[-\,c\right] K^* \left[\frac{E}{E_d} + 1\right],$$

$$1 - K^* = \int_0^\infty \exp\left[-\,y - E_1(y)\right] \int_0^1 \exp\left[-\,sy\right] l[E_d s]\,\mathrm{d}s\,\mathrm{d}y\,,$$

and from this we obtain the trivial result that $K^* \geqslant K$ which shows that a more thorough consideration of the quantitative form of $l(E')$ is of no avail.

(*) It is to be noted that while (8) is merely an asymptotical expression, it yields a fairly good approximation already for $E \doteq 3E_d$. This has been shown by SEITZ and KÖHLER for a special case of (1). Incidentally it can be shown that the difference $m_1(k) - e^{-c} K\big((E/E_d) + 1\big)$ for $E \to +\infty$ tends towards zero better than $\exp\left[-\,c_1 E\right]$, $c_1 > 0$.

Our original purpose, namely the decreasing of $m_1(E)$ by a factor five, could not be therefore completely achieved. It has been revealed however, which form of $l(E')$ could accomplish that. Let be e.g.

$$(9) \qquad l(E') = \exp\left|\ \alpha\ \frac{E}{E_d}\right|. \qquad\qquad 0 \cdot E' \quad \gg ,$$

where α is some suitable positive parameter. It is obvious that a replacement function of this type is more consistent with our general ideas in dynamic processes in solids.

It follows from (2) that the average number is given now by the equation (*)

$$(10) \quad Em_1(E) = \int_0^E m_1(E')\,dE' + \Delta(E - E_d)\int_0^{E-E_d}\left(1 - \exp\left|\ -\ \alpha\ \frac{E'}{E_d}\right|\right) m_1(E')\,dE'\ .$$

In order to slove (10) let us introduce the Laplace transform of $m_1(E)$, that is

$$\int_0^\infty \exp\left[-\ pE\right]m_1(E)\,dE = n_1(p)\ ,$$

and from (10) we obtain

$$(11) \qquad -n_1'(p) = \frac{1 + \exp\left[-\ pE_d\right]}{p}\,n_1(p) - \frac{\exp\left[\ pE_d\right]}{p}\,n_1\left(p - \frac{\alpha}{E_d}\right).$$

Since $n_1(p+\alpha/E_d)$ with $0 \leqslant p < \infty$ is regular, it is convenient to consider it an « inhomogenous » term in eq. (11). Since now (11) in this way for the boundary condition $\lim_{p\to\infty} pn_1(p) = 1$, we find

$$(12) \qquad n_1(p) = \frac{\exp\left[E_1(pE_d)\right]}{p}\left\{1 - \int_{pE_d}^\infty \exp\left[-\ y - E_i(y)\right]n_1\left(\frac{y + \alpha}{E_d}\right)\,dy\right\}.$$

(*) In this case the transformation $G(p;x)$ satisfies the non linear retarded differential equation

$$G'(p) = \frac{1 - \exp\left[-\ pE_d\right]}{p}\,G(p) + \exp\left[-pE_d\right]G(p)\left|\,G(p) - G\left(p + \frac{\alpha}{E_d}\right) + \frac{1}{p + \alpha/E_d}\right|\ .$$

The simple use of this equation leads only to the proof of the existence of the asymptotical expression of the moments. In the explicit determination of the asymptotical expression is required, then for each moment a particular ad hoc method is to be desired.

Hence, for $p \to 0$, we have

$$n_1(p) \sim K(\alpha) \left[\frac{1}{p^2 E_d} + \frac{1}{p} \right],$$

and upon retransformation we obtain

(13)
$$m_1(E) \sim K(\alpha) \left[\frac{E}{E_d} + 1 \right],$$

where

$$K(\alpha) = \exp\left[- c\right] \left\{ 1 - \int_0^\infty \exp\left[- y - E_1(y)\right] n_1 \left(\frac{y + \alpha}{E_d} \right) dy \right\}.$$

that is, $K(\alpha)$ is a positive constant depending only on α. This form of $K(\alpha)$ serves only to prove its existence since $n_1(y)$ is unknown. Nevertheless, even if the explicit value of $K(\alpha)$ cannot be given, its asymptotical expansion for $\alpha \to 0$ can be calculated:

(14)
$$K(\alpha) = e^c \alpha + a_1 \alpha^2 \ln \frac{1}{\alpha} + a_2 \alpha^2 + \dots .$$

The derivation of (14) is rather lengthy. For illustration we shall give in the Appendix only the proof of the relation $\lim_{\alpha \to 0} \left(K(\alpha)/\alpha \right) = e^c$, ($c = 0,577 =$ the Euler constant).

APPENDIX

Let be

$$n_1\left(\frac{y}{E_d} \right) = \Phi(y),$$

then we obtain from (11)

$$- \Phi'(y) = \left\{ 1 + \frac{\exp\left[- y\right]}{y} \right\} \Phi(y) - y^{-1} \exp\left[- y\right] \Phi(y + \alpha).$$

It can be seen that $\Phi(y) = \Phi(y, \alpha)$ is a regular function of the parameter α for $y \geqslant \delta > 0$ thus it is possible to write

$$\Phi(y) = \sum_0^\infty \frac{\alpha^n}{n} \Phi_n(y),$$

where $\Phi_n(y)$ does not depend any more on α.

By easy calculation (comparing to the coefficients) it becomes apparent that $\Phi_0(y) = y^{-1}$ and

$$(15) \qquad \Phi_n(y) = -\sum_{l=0}^{n-1} \binom{n}{l} \frac{1}{y} \int_y^\infty \exp\left[-x\right] \Phi_l^{(n-1)}(x)\,\mathrm{d}x\,, \qquad n \geqslant 1\,.$$

For a special case

$$\Phi_1(y) = y^{-2} \exp\left[-y\right] - y^{-1} E_1(y)\,.$$

It is seen by complete induction that

$$(16) \qquad \Phi_n(y) \sim c_n y^{-n-1}\,, \qquad n \geqslant 1\,.$$

Let us now consider the expression for

$$[17] \quad K(\alpha) = \exp\left[-c\right] - \sum_{n=0}^\infty \frac{\alpha_n}{n!} \exp\left[-c\right] \int_0^\infty \Phi_n(y+\alpha) \exp\left[-y - E_1(y)\right]\mathrm{d}y\,.$$

By a simple calculation we find

$$(18) \qquad S = \lim_{\alpha \to 0} \frac{K(\alpha)}{\alpha} = 1 + \int_0^\infty E_1(y) \exp\left[-y - \int_0^y \frac{1 - \exp\left[-z\right]}{z}\,\mathrm{d}z\right]\,\mathrm{d}y +$$

$$+ \int_0^\infty \frac{1 - \exp\left[-y\right]}{y} \exp\left[-y - \int_0^y \frac{1 - \exp\left[-z\right]}{z}\,\mathrm{d}z\right]\mathrm{d}y - \sum_{n=2}^\infty \frac{p_n}{n!}\,,$$

$$p_n = \exp\left[-c\right] \lim_{\alpha = 0} \alpha^{n-1} \int_0^\infty \Phi_n(y+\alpha) \exp\left[-y - E_1(y)\right]\mathrm{d}y\,.$$

On the basis of a Tauber theorem from (16) we readily get that $p_n = c_n n^{-1}(n-1)^{-1}$. The integrals in (18) can be evaluated trivially so that we obtain the formula

$$(19) \qquad S = 2 - \sum_{n=2}^\infty \frac{c_n}{n(n-1)n!}\,.$$

Making use of (15) for the terms in c_n we can write the recursion formula

$$(20) \qquad c_n = -\sum_{l=1}^{n-1} \binom{n}{l} \frac{(n-1)!}{l!} (-1)^{n-l} c_l\,, \qquad n \geqslant 1\,, \quad c_0 = 1\,.$$

Introducing now the « strong » generating function

$$G(t) = \sum_{n=0}^{\infty} \frac{c_n}{(n!)^2} t^n .$$

$G(t)$ can be evaluated from the equation

$$tG''(t) - G(t) = - \exp[-t] G(t) .$$

Thus

$$G(t) = \exp\left[\int_0^t \frac{1 - \exp[-z]}{z} dz \right] .$$

Now the sum in the expression for S can be written in the form of an integral:

$$-\sum_{n=2}^{\infty} \frac{c_n}{n(n-1)n!} = \int_0^{\infty} \frac{1 + t - G(t)}{t^2} \exp[-t] dt .$$

which can be evaluated with the knowledge of $G(t)$. Its value is found to be $e^c - 2$, consequently $S = e^c$.

It is possible to make an estimate as to the correct value of α. From the eq. (14) the root of

$$K(\alpha) = \frac{\exp[-c]}{5} ,$$

is given as $\alpha \sim 0.09 \sim 1/11$.

It is difficult to present a theoretical interpretation of a replacement function with such a small α as parameter. It seems rather improbable. Although it is true that the replacement is not the most important effect involved in the process, it looks like an exaggeration to obtain the reduction by a factor of 5 with the previous theory unchanged and replacement criterions introduced. Nevertheless, it is apparent that the consideration of the replacement cannot be avoided if better agreement with experiment is intended.
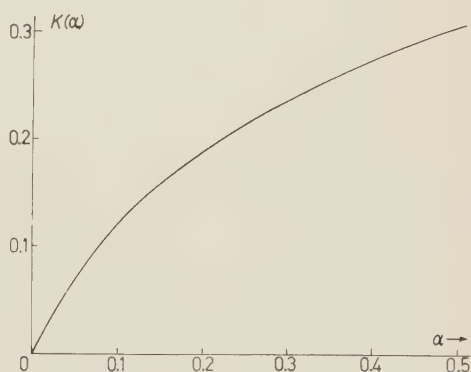
Fig. 1. – Function $K(\alpha)$ which has been determined by numerical integration from (10) for fixed α as the asymptotic value of the expression $m_1(E)/((E/E_d)+1)$, $E \to \infty$.

* * *

RIASSUNTO (*)

Si fa il calcolo del numero di danneggiamenti del reticolo dovuti all'irradiazione con particelle di alta energia. Basandosi sull'ipotesi avanzata in ([7]) che gli atomi causa degli spostamenti dovuti a collisione siano catturati nei posti competenti nel reticolo agli atomi spostati, si scrive la probabilità di cattura nella forma esponenziale $\exp\left[-\alpha(E/E_d)\right]$ e si ottiene il numero medio dei danneggiamenti del reticolo: $m_z(E) \sim K(z)((E/E_d)-1)$.

Méthodes d'approximation pour la détermination de potentiels semi-phénoménologiques nucléon-nucléon.

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Summary. — We wish to construct a formalism allowing to connect phenomenological parameters of different potentials in nucleon-nucleon scattering with experimental results: phase-shifts and coupling parameters. We show that phase-shifts are obtained by numerical calculations of two integrals, the two integrants being solutions of Volterra's coupled integral equations whose kernels explicitly depend on potentials.

Introduction.

Les progrès réalisés et les nombreuses études portant sur l'analyse en déphasage de la diffusion nucléon-nucléon jusqu'à 400 MeV permettent d'espérer dans un avenir proche une connaissance non ambiguë de ces déphasages. On peut donc prendre ces déphasages expérimentaux comme point de départ et essayer de les relier d'une façon pratique avec une famille de potentiels consistante avec les considérations théoriques. Ces considérations aussi bien que les données de l'expérience nous astreignent à inclure l'O.P.E.P. Nous tiendrons compte aussi d'un cœur dur de rayon $r_c \neq 0$. Une méthode d'itération particulière obtenue par A. Martin pour $l = 0$ [1] et qui fournit des approximations commodes s'étend de diverses manières pour $l \neq 0$ [2-5] et pour une large classe de potentiels [5] comprenant entre autres $\sum \exp[-n\mu r]r^{-p}$.

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[1] A. Martin: *Nuovo Cimento*, **14**, 403 (1959).

[2] A. Martin: *Nuovo Cimento*, **15**, 99 (1960).

[3] De Alfaro et C. Rossetti: *Nuovo Cimento*, **18**, 780 (1960).

[4] L. Bertocchi, C. Ceolin et N. Tonin: *Nuovo Cimento*, **18**, 770 (1960).

[5] H. Cornille: *Compt. Rend. Acad. Sci.*, **251**, 2135, 2308 (1960).

1. – Potentiels semi-phénoménologiques.

Nous considérons ([6]) pour la diffusion proton-proton la famille de potentiels (1) s'accordant avec les considérations habituelles d'invariance; pour le système neutron-proton un terme additionnel similaire à (1) doit être ajouté correspondant à l'isospin $I = 0$,

$$(1) \qquad V = V_c + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 V_\sigma + V_{LS} \boldsymbol{LS} + V_T S_{12} + \tfrac{1}{2} V_Q \big((\boldsymbol{L} \cdot \boldsymbol{\sigma}_1)(\boldsymbol{L} \cdot \boldsymbol{\sigma}_2) + (\boldsymbol{L} \cdot \boldsymbol{\sigma}_2)(\boldsymbol{L} \cdot \boldsymbol{\sigma}_1) \big) .$$

Les potentiels semi-phénoménologiques auront les caractères habituels: cœur dur, appartenance à la famille (1) le terme dominant à grande distance étant l'O.P.E.P. ([2])

$$(2) \qquad \frac{f^2}{4\pi} \frac{1}{3} (\boldsymbol{I}_1 \boldsymbol{I}_2) \left(\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 + S_{12} \left(1 + \frac{3}{\mu r} + \frac{3}{\mu^2 r^2} \right) \right) \frac{\exp [-\mu r]}{\mu r} .$$

Nous supposons que les divers V_i appartiennent à la famille (3) (p_0 étant réel > 0 et fini)

$$(3) \qquad V_i(r) = \exp [-\mu r] \int_0^\infty C_i(\gamma) \exp [-\gamma r] \mathrm{d}\gamma , \qquad |C_i(\gamma)| < M_i \gamma^{p_0} .$$

En Section 2 nous étudions le cas des équations non couplées (singulet, triplet $L = J$), en Section 3 celui des équations couplées avec $r_c \neq 0$ et nous montrons la consistence de notre formalisme pour $r_c = 0$.

2. – Equations non couplées.

L'équation de Schrödinger s'écrit en posant la variable radiale égale à $r_c + r$ où $r > 0$

$$(4) \qquad \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + K^2 - \frac{l(l+1)}{(r_c + r)^2} \right) u_l(r_c + r) = V(r_c + r) u_e(r_o + r) ,$$

V étant une combinaison linéaire de potentiels de la famille (3) où nous avons omis la dépendance en l. Pour définir la matrice S nous utilisons une méthode donnée ([3]) dans le cas où $r_c = 0$.

Soient 2 solutions de (4)

$$(5) \qquad R_l(r_c + r, \pm K) = \int_0^\infty h_l((\pm K + i\alpha)(r_c + r)) \varrho_{\pm K}(\alpha) \mathrm{d}\alpha \underset{r \to \infty}{\simeq} h_l(\pm K(r_c + r)) ,$$

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([6]) S. OKUBO et R. E. MARSHAK: _Ann. Phys._, **4**, 166 (1958).

où

$$(6) \qquad h_l(x) = (-i)^{l+1} \exp\left[iKx\right] \sum_{p=0}^{l} \left(\frac{i}{2x}\right)^p a_{l,p} \qquad \text{avec} \quad a_{l,p} = \frac{(l+p)!}{p!\,(l-p)!}\,,$$

soit une solution $u_l(r_c + r)$ telle que

$$u_l(r_c + r) = R_l(-K, r_c + r) - (-1)^l S_l(K)\, R_l(K, r_c + r)$$

la condition $u_l(r_c) = 0$ nous donne $S_l(K)$

$$(7) \qquad S_l(K) = (-1)^l \frac{R_l(-K, r_c)}{R_l(K, r_c)}\,.$$

La détermination de $x_{\pm K}(\alpha) = \exp\left[-\alpha r_c\right]\varrho_{\pm K}(\alpha)$ se fait par la résolution d'une équation de Volterra,

$$(8) \qquad x_K(\alpha) = \delta(\alpha) + B(\alpha, K)\int_0^{\alpha-\mu} G_l(\alpha, \beta, K)x_K(\beta)\,\mathrm{d}\beta,$$

avec $B(\alpha, K) = \big(\alpha(\alpha - 2iK)\big)^{-1}$ et $G_l \equiv 0$ pour $\beta > \alpha - \mu$ est donné par:

$$(9) \qquad G_l(\alpha, \beta, K) = \exp\left[-(\alpha-\beta)r_c\right]\int_0^{\alpha-\beta-\mu} \mathscr{K}_l^l(\alpha, \beta, \gamma+\mu, K)\, C(\gamma)\,\mathrm{d}\gamma,$$

$$\exp\left[-(\gamma+\mu)(r_c+r)\right]h_l\big((K+i\beta)(r_c+r)\big) = \int_0^{\infty} h_l\big((K+i\alpha)(r_c+r)\big)\cdot$$

$$\cdot\mathscr{K}_l^l(\alpha, \beta, \gamma+\mu, K)\,\mathrm{d}\alpha,$$

$\mathscr{K}_l^l$ est $\equiv 0$ pour $\alpha < \beta + \gamma + \mu$, son expression et sa détermination sont données en Appendice A.

En Appendice B nous montrons que pour K réel $|x_K(\alpha)| < |B(\alpha, K)|$ l'existence de $R_l(K, r_c)$ en résulte.

$\qquad$ Soit $f_l\big((K+i\alpha)r_c\big) = (i)^{l+1}\exp\left[\alpha r_c\right]\exp\left[-iKr_c\right]h_l\big((K+i\alpha)r_c\big)$ d'où:

$$S_l'(K) = (-1)^l S_l(K)\exp\left[2iKr_c\right] = \frac{\mathscr{R}_l(-K, r_c)}{\mathscr{R}_l(K, r_c)}\,,$$

où

$$(11) \qquad \mathscr{R}_l(K, r_c) = f_l(K, r_c) + \int_{\mu}^{\infty} f_l\big((K+i\alpha)r_c\big)x_K(\alpha)\,\mathrm{d}\alpha,$$

comme

$$\mathscr{K}_1^l(\alpha, \beta, \gamma + \mu, K) = (\mathscr{K}_1^l(\alpha, \beta, \gamma + \mu, -K^*))^*$$

on déduit

$$\mathscr{R}_l^*(K) = \mathscr{R}_l(-K^*) \, .$$

On a donc d'une part les propriétés habituelles

$$S_l(K) = (S_l(-K))^{-1}, \qquad S_l(K^*) = (S_l^*(K))^{-1}, \qquad S_l(-K^*) = S_l^*(K) \, ,$$

d'autre part si K est réel $\mathscr{R}_l(K)$ a pour phase $-(\delta + K r_c + l(\pi/2))$.

On déduit de (8) une équation intégrale semblable pour $f_l((K+i\alpha)r_c)x_K(\alpha) = a_K(\alpha) - i b_K(\alpha)$. Pour K réel, les fonctions a_K et b_K vérifient le système réel couplé (12)

$$(12) \qquad \begin{pmatrix} a_K(\alpha) \\ b_K(\alpha) \end{pmatrix} = \delta(\alpha) \begin{pmatrix} a_{0,K} \\ b_{0,K} \end{pmatrix} + \int_0^{\alpha-\mu} \begin{pmatrix} \operatorname{Re} g & \operatorname{Im} g \\ -\operatorname{Im} g & \operatorname{Re} g \end{pmatrix} \begin{pmatrix} a_K(\beta) \\ b_K(\beta) \end{pmatrix} .$$

où $a_{0K} - i b_{0K} = f_l(Kr_c)$ et $\operatorname{Re} g$, $\operatorname{Im} g$ sont les parties réelles et imaginaires de

$$g(\alpha, \beta, K) = B(\alpha, K) G_l(\alpha, \beta, K) f_l((K + i\alpha)r_c)(f_l((K + i\beta)r_c))^{-1} \, .$$

Ces quantités se déterminent aisément si on remarque que pour K réel

$$g(\alpha, \beta, K) = g^*(\alpha, \beta, -K) \, ; \qquad f_l(\alpha, \beta, K) = f_l^*(\alpha, \beta, -K) \, ,$$

finalement:

$$(13) \qquad \operatorname{tg}\left(\delta + K r_c + l\,\frac{\pi}{2}\right) = \frac{\displaystyle\int_0^\infty b(\alpha)\,\mathrm{d}\alpha}{\displaystyle\int_0^\infty a(\alpha)\,\mathrm{d}\alpha} \, .$$

Le cas $r_c = 0$ a été étudié [2]; pour une transformation du type (5), il est clair que la méthode et les résultats donnés pour $\exp[-\mu r]r^{-1}$ par ALFARO et ROSSETTI [3] s'appliquent pour $|C(\gamma)| < C^{re}\gamma^{\nu-1}$ $(0 < \nu < 2)$.

3. – Equations couplées.

$$(14) \qquad \begin{pmatrix} \dfrac{\mathrm{d}^2 u_j}{\mathrm{d}r^2} \\[2mm] \dfrac{\mathrm{d}^2 w_j}{\mathrm{d}r^2} \end{pmatrix} +$$

$$+ \begin{pmatrix} K^2 - \dfrac{J(J-1)}{(r_c+r)^2} + V_1(r_c+r) & V_T(r_c+r) \\[3mm] V_T(r_c+r) & K^2 - \dfrac{(J+1)(J+2)}{(r_c+r)^2} + V_2(r_c+r) \end{pmatrix} \begin{pmatrix} u_j \\ w_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Nous avons omis la dépendance en J des divers V_i appartenant à la famille (3). Nous désirons relier simplement les potentiels et les données expérimentales; les déphasages propres δ_x, δ_β et le paramètre de mélange ε de BLATT et BIEDENHARN [7] sont supposés connus, nous ne faisons nos raisonnement que sur un seul état propre (α) de la matrice S.

Nous supposons la condition asymptotique [*]

$$(15) \qquad \begin{pmatrix} u_J \\ w_J \end{pmatrix} \simeq \begin{pmatrix} \cos \varepsilon [h_{J-1} (-K(r+r_c)) - (-1)^{J-1} \exp[2i\delta]h_{J-1}(K(r+r_c))] \\ \sin \varepsilon [h_{J+1} (-K(r+r_c)) - (-1)^{J+1} \exp[2i\delta]h_{J+1}(K(r+r_c))] \end{pmatrix}.$$

Nous définissons 2 couples de solutions de (14)

$$(16) \qquad \begin{pmatrix} R(\pm K, r_c + r) \\ T(\pm K, r_c + r) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_0^\infty h_{J-1} ((\pm K + i\alpha)(r_c + r)) \varrho_{\pm K}(\alpha)\,d\alpha \\ \displaystyle\int_0^\infty h_{J+1} ((\pm K + i\alpha)(r_c + r)) \tau_{\pm K}(\alpha)\,d\alpha \end{pmatrix},$$

$$(17) \qquad \underset{r \to \infty}{\simeq} \begin{pmatrix} \cos \varepsilon h_{J-1}(\pm K(r_c + r)) \\ \sin \varepsilon h_{J+1}(\pm K(r_c + r)) \end{pmatrix}.$$

Soit une solution

$$(18) \qquad \begin{pmatrix} u_J(r_c + r) \\ w_J(r_c + r) \end{pmatrix} = \begin{pmatrix} R(-K, r_c + r) - (-1)^{J-1} \exp[2i\delta]R(K, r_c + r) \\ T(-K, r_c + r) - (-1)^{J-1} \exp[2i\delta]T(K, r_c + r) \end{pmatrix},$$

la condition $u_J(r_c) = w_J(r_c) = 0$ donne

$$(19) \qquad \exp[2i\delta](-1)^{J-1} = \frac{R(-K, r_c)}{R(K, r_c)} = \frac{T(-K, r_c)}{T(K, r_c)}.$$

$\varrho_{\pm K}(\alpha) \exp[-\alpha r_c] = x_{\pm K}(\alpha)$, $\tau_{\pm K}(\alpha) \exp[-\alpha r_c] = y_{\pm K}(\alpha)$ satisfont à une équation intégrale couplée de Volterra qui se déduit aisément en portant les solu-

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[7] J. M. BLATT et L. C. BIEDENHARM: *Phys. Rev.*, **86**, 399 (1952).

[*] On pourrait aussi imposer

$$\begin{pmatrix} u_J \\ w_J \end{pmatrix} \simeq \begin{pmatrix} \cos \varepsilon [\exp[-i(K(r_c+r) - (J-1)\pi/2)] - \exp[i(2\delta + K(r_c+r) - (J-1)\pi/2)]] \\ \sin \varepsilon [\exp[-i(K(r_c+r) - (J+1)\pi/2)] - \exp[i(2\delta + K(r_c+r) - (J+1)\pi/2)]] \end{pmatrix}$$

on obtiendrait des résultats semblables: il faudrait modifier les conditions (17) et (23).

tions (16) dans (14) avec la condition (17)

$$(20) \quad \begin{pmatrix} x_K(\alpha) \\ y_K(\alpha) \end{pmatrix} = \begin{pmatrix} x_{0,K}(\alpha) \\ y_{0,K}(\alpha) \end{pmatrix} + B(\alpha, K) \int_0^{\alpha-\mu} \begin{pmatrix} G_{J-1}^{J-1}(\alpha, \beta, K), G_{J-1}^{J+1}(\alpha, \beta, K) \\ G_{J+1}^{J-1}(\alpha, \beta, K), G_{J+1}^{J+1}(\alpha, \beta, K) \end{pmatrix} \begin{pmatrix} x_K(\beta) \\ y_K(\beta) \end{pmatrix} d\beta ,$$

où $G_{J+\eta}^{J\pm\eta}$, $(\eta = \pm 1)$ nul pour $\alpha < \beta + \mu$ est donné par

$$(21) \quad G_{J+\eta}^{J\pm\eta}(\alpha, \beta, K) = \exp\left[-(\alpha-\beta)r_c\right] \int_0^{\alpha-\beta-\mu} C_{+\eta}^{\pm\eta}(\gamma) \mathscr{K}_{J+\eta}^{J\pm\eta}(\alpha, \beta, \gamma+\mu, K) d\gamma ,$$

$$(22) \quad \exp\left[-(\gamma+\mu)(r_c+r)\right] h_{J\pm\eta}((K+i\beta)(r_c+r)) =$$

$$= \int_0^\infty \mathscr{K}_{J+\eta}^{J\pm\eta}(\alpha, \beta, \gamma+\mu, K) h_{J+\eta}((K+i\alpha)(r_c+r)) d\alpha , \quad (J = 1, 2, \dots) .$$

Nous avons pour conditions initiales en α suivant la condition (17)

$$(23) \quad \begin{pmatrix} x_{0,K}(\alpha) \\ y_{0,K}(\alpha) \end{pmatrix} = \begin{pmatrix} x_0 \delta(\alpha) \\ y_0 \delta(\alpha) \end{pmatrix}, \quad x_0 = \cos\varepsilon , \quad y_0 = \sin\varepsilon .$$

Les expressions et déterminations des noyaux $\mathscr{K}_{J+\eta}^{J\pm\eta}$ sont données en Appendice A. En Appendice B nous montrons que pour les conditions initiales (23) et K réel

$$\begin{pmatrix} |x_K(\alpha)| \\ |y_K(\alpha)| \end{pmatrix} \underset{\alpha\to\infty}{<} \begin{pmatrix} |B(\alpha, K)| \\ |B(\alpha, K)| \end{pmatrix},$$

l'existence de $R(K, r_c)$ et $T(K, r_c)$ en résulte. Comme dans le cas des équations non couplées si on pose

$$(24) \quad \begin{pmatrix} \mathscr{R}(K, r_c) \\ \mathscr{T}(K, r_c) \end{pmatrix} = \begin{pmatrix} \cos\varepsilon f_{J-1}(Kr_c) + \int_\mu^\infty f_{J-1}((K+i\alpha)r_c) x_K(\alpha) d\alpha \\ \\ \sin\varepsilon f_{J+1}(Kr_c) + \int_\mu^\infty f_{J+1}((K+i\alpha)r_c) y_K(\alpha) d\alpha \end{pmatrix},$$

$$(25) \quad \exp\left[2i\left(\delta + Kr_c + (J-1)\frac{\pi}{2}\right)\right] = \frac{\mathscr{R}(-K, r_c)}{\mathscr{R}(K, r_c)} = \frac{\mathscr{T}(-K, r_c)}{\mathscr{T}(K, r_c)}, \quad (K \text{ réel}) .$$

Etant donné que

$$\mathscr{K}_{J+\eta}^{J\pm\eta}(K) = \left(\mathscr{K}_{J+\eta}^{J\pm\eta}(-K^*)\right)^* \quad \text{et que} \quad f_J((K+i\alpha)r_c) = \left(f_J((-K^* + i\alpha)r_c)\right)^*$$

on voit aisément que pour K réel $\mathscr{R}(K, r_c)$ et $\mathscr{T}(K, r_c)$ ont pour phase $-(\delta + Kr_c + (J-1)\pi/2)$.

On déduit de (20) un système intégral semblable pour

$$\begin{pmatrix} f_{J-1}((K + i\alpha)r_c)x_K(\alpha) = a_K(\alpha) - ib_K(\alpha) \\ f_{J+1}((K + i\alpha)r_c)y_K(\alpha) = c_K(\alpha) - id_K(\alpha) \end{pmatrix},$$

(a, b, c, d) vérifiant un système couplé réel de Volterra

$$(26) \quad \begin{pmatrix} a(\alpha) \\ b(\alpha) \\ c(\alpha) \\ d(\alpha) \end{pmatrix} = \delta(\alpha) \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{pmatrix} +$$

$$+ \int_0^\alpha \begin{pmatrix} \mathrm{Re}\, g_{-1}^{-1} & \mathrm{Im}\, g_{-1}^{-1} & \mathrm{Re}\, g_{-1}^{+1} & \mathrm{Im}\, g_{-1}^{+1} \\ -\mathrm{Im}\, g_{-1}^{-1} & \mathrm{Re}\, g_{-1}^{-1} & -\mathrm{Im}\, g_{-1}^{+1} & \mathrm{Re}\, g_{-1}^{+1} \\ \mathrm{Re}\, g_{+1}^{-1} & \mathrm{Im}\, g_{+1}^{-1} & \mathrm{Re}\, g_{+1}^{+1} & \mathrm{Im}\, g_{+1}^{+1} \\ -\mathrm{Im}\, g_{+1}^{-1} & \mathrm{Re}\, g_{+1}^{-1} & -\mathrm{Im}\, g_{+1}^{+1} & \mathrm{Re}\, g_{+1}^{+1} \end{pmatrix} \begin{pmatrix} a(\beta) \\ b(\beta) \\ c(\beta) \\ d(\beta) \end{pmatrix} \mathrm{d}\beta,$$

où $a_0 - ib_0 = \cos\varepsilon\, f_{J-1}(Kr_c)$, $c_0 - id_0 = \sin\varepsilon\, f_{J+1}(Kr_c)$; $\mathrm{Re}\, g_{+\eta}^{\pm\eta}$ et $\mathrm{Im}\, g_{+\eta}^{\pm\eta}$ sont les parties réelles et imaginaires de

$$g_{J+\eta}^{J\pm\eta}(\alpha, \beta, K) = B(\alpha, K)G_{J+\eta}^{J\pm\eta}(\alpha, \beta, K)f_{J+\eta}((K + i\alpha)r_c)\big(f_{J\pm\eta}((K + i\beta)r_c)\big)^{-1}.$$

Ici encore ces quantités se déterminent aisément car pour K réel

$$g_{J+\eta}^{J\pm\eta}(K) = \big(g_{J+\eta}^{J\mp\eta}(-K)\big)^*.$$

Enfin:

$$(27) \quad \mathrm{tg}\left(\delta + Kr_c + (j-1)\frac{\pi}{2}\right) = \frac{\int_0^\infty b(\alpha)\,\mathrm{d}\alpha}{\int_0^\infty a(\alpha)\,\mathrm{d}\alpha} = \frac{\int_0^\infty d(\alpha)\,\mathrm{d}\alpha}{\int_0^\infty c(\alpha)\,\mathrm{d}\alpha}.$$

Le système (26) se résout par itération, les n premières itérations donnant la solution exacte pour $\alpha \in [0, n\mu]$ pour $\alpha \in [0, \mu]$ on peut imposer aux $C_i(\alpha)$ de s'identifier avec l'O.P.E.P. et utiliser des paramètres ajustables pour les termes correspondant à l'échange de plus d'un pion.

Remarque: Si nous voulions définir un formalisme déterminant δ et ε ou si nous ne considérions que δ comme donnée expérimentale, il faudrait comme

R. Vinh-Mau et A. Martin ([8]) prendre une base complète de l'éq. (14) par exemple

$$\begin{pmatrix} R_\lambda(\pm K, r_c + r) \\ T_\lambda(\pm K, r_c + r) \end{pmatrix} \underset{r \to \infty}{\simeq} \begin{pmatrix} h_{J-1}(\pm K(r_c + r)) \\ D_\lambda h_{J+1}(\pm K, (r_c + r)) \end{pmatrix}, \quad \lambda = 1, 2. \quad D_1 = 1. \quad D_2 = -1.$$

Une solution quelconque, combinaison linéaire des 4 solutions de la base, soumise aux conditions asymptotiques (15) ne dépend plus que de 2 paramètres liés à δ et ε que l'on peut déterminer en imposant $u(r_c) = w(r_c) = 0$. L'existence des $\begin{pmatrix} R_\lambda(\pm K, r_c) \\ T_\lambda(\pm K, r_c) \end{pmatrix}$ sous forme de transformés du type (16) est assurée pour K réel (Appendice B : $X_0 = 1$, $Y_0 = \pm 1$).

L'étude de ces fonctions pour K complexe permettrait d'obtenir les propriétés d'analyticité de la matrice S dans le cas d'un cœur dur et de potentiels du type (3).

Cas de $r_c = 0$. – Si on suppose $V_T(r) \underset{r \to 0}{\simeq} V_{T_0} r^{-\nu}$, ($V_{T_0} = C^{re}$) et les autres potentiels de la forme $V_i(r) \underset{r \to 0}{\simeq} C^{re} r^{-\xi_i}$ on trouve comme système fondamental de solutions de l'éq. (14) au voisinage de l'origine pour $\nu < 2$ et $\xi_i < 2$

$$\begin{pmatrix} u \\ w \end{pmatrix} \underset{r \to 0}{\simeq} \begin{pmatrix} r^J \varphi_1(r) \\ r^{J+2-\nu} \psi_1(r) \end{pmatrix} + \begin{pmatrix} r^{J+1-\nu} q_2(r) \\ r^{J+2} \psi_2(r) \end{pmatrix} + \begin{pmatrix} r^{-(J-1)} q_3(r) \\ r^{-(J-3+\nu)} \psi_3(r) \end{pmatrix} - \begin{pmatrix} r^{-(J-1+\nu)} q_4(r) \\ r^{-(J+1)} \psi_4(r) \end{pmatrix}.$$

$\varphi_i(0)$ et $\psi_i(0)$ étant des C^{tes}; $\psi_i(0) = \lambda_i q_i(0)$, λ_i ne dépendant que des coefficients de (14) et de V_{T_0}; si $\nu = 0$ le dernier terme en u devient $r^{-(J-1)} L_i q_4(r)$.

On étudie le cas $0 < \nu < 2$. Nous définissons toujours 2 solutions du type (16) astreintes aux conditions (17), le système intégral (20) est valable pour $\varrho_K(\alpha)$, $\tau_K(\alpha)$ la condition de régularité (18) est maintenant remplacée par

$$(29) \qquad \exp[2i\delta](-1)^{J-1} = \frac{\lim\limits_{r \to 0} r^{J-1+\nu} R(-K, r)}{\lim\limits_{r \to 0} r^{J-1+\nu} R(K, r)} = \frac{\lim\limits_{r \to 0} r^{J+1} T(-K, r)}{\lim\limits_{r \to 0} r^{J+1} T(K, r)}.$$

Nous justifions l'existence de ces limites en Appendice B pour la famille

$$(30) \qquad |C_T(\gamma)| < M_T \gamma^{\nu-1} \quad \text{et} \quad |C_i(\gamma)| < M_i \gamma^{\xi_i-1} \qquad (0 < \xi_i < 2)$$

en montrant que $x_K(\alpha) = \varrho_K(\alpha) \big(K/(K+i\alpha)\big)^{J-1} \underset{\alpha \to \infty}{\simeq} \alpha^{\nu-1}$

$$(31) \qquad y_K(\alpha) = \tau_K^*(\alpha) \left(\frac{K}{K+i\alpha}\right)^{J+1} \underset{\alpha \to \infty}{\simeq} \alpha^{-(\xi+1)},$$

([8]) R. Vinh-Mau et A. Martin *Nuovo Cimento*. **20**, 390 (1961), indépendamment ont étudié les équations couplées pour obtenir les propriétés d'analyticité de la matrice S.

(ζ est le plus petit de $2-\nu$, $2-\xi_i$) $\begin{pmatrix} x_K(\alpha) \\ y_K(\alpha) \end{pmatrix}$ vérifiant un système intégral du type (20) où

$$(32) \qquad G_{J+\eta}^{J\pm\eta}(\alpha,\beta,\mu,K) = \frac{(K+i\beta)^{J\pm\eta}}{(K+i\alpha)^{J-\eta}} \int_0^{\cdot\beta\cdot\mu} C_{+\eta}^{\pm\eta}(\gamma)\,\mathcal{K}_{J+\eta}^{J\pm\eta}(\alpha,\beta,\gamma+\mu,K)\,\mathrm{d}\gamma \;.$$

* * *

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APPENDICE A

Determination et calcul des noyaux $\mathcal{K}_{J+\eta}^{J\pm\eta}$.

a) Il a été proposé 2 méthodes pour déterminer $\mathcal{K}_i^l$ [3-5]. Nous rappelons [5] qu'étant donnée la propriété

$$\frac{\partial}{\partial r}\left[(r_c+r)^l h_l(K(r_c+r))\right] = K(r_c+r)^l h_{l-1}\big(K(r_c+r)\big)\,,$$

la transformation pour $f(r_c+r)$ quelconque

$$f(r_c+r) = \int_0^\infty \mathcal{K}_l(\alpha,K) h_l\big((K+i\alpha)(r_c+r)\big)\,\mathrm{d}\alpha\,,$$

se ramène à

$$i\exp\left[-iK(r_c+r)\right]\left(\frac{1}{(r_c+r)}\frac{\partial}{\partial r}\right)^l\big((r_c+r)^l f(r_c+r)\big) =$$

$$= \int_0^\infty \exp\left[-\alpha r\right]\big[\mathcal{K}_l(\alpha,K)(K+i\alpha)^l\exp\left[-\alpha r_c\right]\big]\mathrm{d}\alpha\;.$$

D'après la propriété de translation des images de Laplace $\mathcal{K}_l$ indépendant de r_c est donné par

$$(A\text{-}1) \qquad (K+i\alpha)^l\mathcal{K}_l(\alpha,K) = \frac{1}{2i\pi}\int_{-i\infty+\tau_0}^{+i\infty+\tau_0}\exp\left[\alpha r\right]\varLambda_l f(r)\,\mathrm{d}r\,,$$

où

$$\varLambda_l f(r) = i\exp\left[-iKr\right]\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^l\big(r^l f(r)\big)\,.$$

Pour l'existence de $\mathscr{K}_i$ nous sommes donc principalement ramenés à l'existence de l'originale dans la transformation de Laplace, l'étude de l'existence de noyaux correspondants à des potentiels quelconques a été exposée en ([5]).

b) D'après ce qui précède

$$(A\text{-}2)\qquad h_{J\pm\eta}((K+i\beta)r)\exp\left[-(\mu+\gamma)r\right]=$$

$$=\int\limits_{0}^{\infty}\mathscr{K}_{J+\eta}^{J\pm\eta}(\alpha,\beta,\gamma+\mu,K)h_{J+\eta}((K+i\alpha)r)\,\mathrm{d}\alpha\,.$$

Nous rappelons aussi que si nous prenons l'originale de (A-2) après multiplication par $\exp\left[-iKr\right]$ la propriété pour $\mathscr{K}_{J+\eta}^{J\pm\eta}$ d'être $\quad 0$ pour $\alpha<\beta+\gamma+\mu$ résulte de ce que $\mathscr{F}(\lambda,\beta,\mu+\gamma)$ est $\quad 0$ pour $\lambda<\beta+\mu-\gamma$ ($\mathscr{F}$ étant l'originale en λ correspondante du 1er membre en r). On peut voir sur (9) et (21) que cette propriété sur $\mathscr{K}$ entraine que $G_{J+\eta}^{J\pm\eta}=0$ pour $\alpha<\beta+\mu$. Enfin cette dernière propriété entraine pour les équations intégrales (8) et (20) que $\begin{pmatrix}x_1(\alpha)\\y_1(\alpha)\end{pmatrix}\equiv 0$ pour $\alpha<\mu$ et par récurrence $\begin{pmatrix}x_n(\alpha)\\y_n(\alpha)\end{pmatrix}\equiv 0$ pour $\alpha<n\mu$.

c) Calcul des noyaux $\mathscr{K}_{J+\eta}^{J\pm\eta}$:

D'après (A-1) et (A-2); $(K+i\alpha)^{J+\eta}\mathscr{K}_{J+\eta}^{J\pm\eta}$ est l'originale de

$$\Delta_{J+\eta}(h_{J\pm\eta}((K+i\beta)r)\exp\left[-(\mu+\gamma)r\right])\,.$$

Pour F quelconque on a

$$(A\text{-}3)\qquad\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{J+\eta}F(r)=\frac{1}{r^{2(J+\eta)}}\left[\sum_{p=1}^{J+\eta}b_{J+\eta,p}r^{p}F^{(p)}\right].$$

Prenant $F=\exp\left[ir\right]$ on obtient $b_{J+\eta,p}=(-\tfrac{1}{2})^{J+\eta-p}a_{J+\eta-1,J+\eta-p}$.
Il suffit donc d'appliquer (A-3) à

$$F=r^{J+\eta}h_{J\pm\eta}((K+i\beta)r)\exp\left[-(\mu+\gamma)r\right],$$

on obtient pour image de Laplace des termes de la forme $\exp\left[-(\beta+\mu+\gamma)r\right]/r^m$; en prenant les originaux on a:

$$(A\text{-}4)\qquad\mathscr{K}_{J+\eta}^{J\pm\eta}(\alpha,\beta,\gamma+\mu,K)=\pm\delta(\alpha-\gamma-\mu-\beta)+$$

$$+\frac{(-i)^{J\pm\eta}}{(k+i\alpha)^{J+\eta}}\theta(\alpha-\gamma-\mu-\beta)\left[\sum_{n=0}^{n_0}\frac{(\alpha-\gamma-\beta-\mu)^n}{n!}\left[\sum_{p=1}^{J+\eta}b_{J+\eta,p}\cdot\right.\right.$$

$$\left.\left.\cdot\left[\sum_{q=0}^{J\pm\eta}C_p^{s-q}(ik-\gamma-\mu-\beta)^{s-q}\left(\frac{i}{2(k+i\beta)}\right)^{J\pm\eta-q}a_{J\pm\eta,J\pm\eta-q}\cdot\Gamma_{q+\eta\mp\eta}^{p+q-s}(p+q-s)!\right]\right]\right],$$

où $n_0 = 2J + \eta \pm \eta - 2$ sauf pour $\mathscr{K}_{J-1}^{J+1}$ où $n_0 = 2J - 1$

$$s = 2J + \eta \pm \eta - 1 - n, \qquad 0 \leqslant s - q \leqslant p \leqslant J + \eta,$$

$$\Gamma_{q+\eta \mp \eta}^{p+q-s}(p + q - s)! =$$

$$= (q + \eta \mp \eta)(q + \eta \mp \eta - 1) \ldots (s + \eta \mp \eta - p + 1) = 1 \qquad \text{si} \quad s = p + q,$$

$$= (q + \eta \mp \eta) \qquad \text{si} \quad s = p + q - 1$$

$$= \ldots,$$

$a_{\nu,\nu'}; b_{\nu,\nu'}$ nuls si $\nu' > \nu$; si $\eta = 0$ on prend $+\delta(\alpha - \gamma - \mu - \beta)$.

Pour $\mathscr{K}_i^l$ on peut utiliser cette formule ou l'expression légérement différente donnée en (^{5}).

APPENDICE B

Comportement des solutions des équations (8) et (20).

On a vu (Appendice A b) que ces solutions sont de la forme

$$\begin{pmatrix} x(\alpha) \\ y(\alpha) \end{pmatrix} = \begin{pmatrix} x_0\delta(x) \\ y_0\delta(\alpha) \end{pmatrix} + \sum_0^\infty \begin{pmatrix} x_m(\alpha) \\ y_m(\alpha) \end{pmatrix} \theta(\alpha - m\mu).$$

Nous voulons montrer que

(B-0)
$$\begin{pmatrix} x(\alpha) \\ y(\alpha) \end{pmatrix}_{\alpha > \mu} = \begin{pmatrix} f_x(\alpha, K) & 0 \\ 0 & f_y(\alpha, K) \end{pmatrix} (A(\alpha, K)),$$

où $(|A(\alpha, K)|)$ borné.

Pour ce faire, nous écrivons ces équations sous la forme

$$\begin{pmatrix} x(\alpha) \\ y(\alpha) \end{pmatrix} = \begin{pmatrix} x_0\delta(\alpha) \\ y_0\delta(\alpha) \end{pmatrix} + (F(\alpha, K)) \int_0^{\alpha-\mu} (\mathscr{H}(\alpha, \beta, K)) \begin{pmatrix} x(\beta) \\ y(\beta) \end{pmatrix} d\beta \quad \text{où}$$

$$F(\alpha, K) = \begin{pmatrix} f_x(\alpha, K) & 0 \\ 0 & f_y(\alpha, K) \end{pmatrix}; \quad \mathscr{H} = (\mathscr{H}_{J+\eta}^{J\pm\eta}); \quad B(G) = (F)(\mathscr{H});$$

les significations de f_x, f_y et $\mathscr{H}$ seront precisées ci dessous.

Supposons qu'il existe m_0 tel que

$$\begin{pmatrix} x_{m_0}(\alpha) \\ y_{m_0}(\alpha) \end{pmatrix} = (F(\alpha, K)) \begin{pmatrix} x_{m_0}(\alpha, K) \\ y_{m_0}(\alpha, K) \end{pmatrix} \theta(\alpha - m_0\mu) \quad \text{où} \quad \begin{pmatrix} |x_{m_0}(\alpha, K)| < X_{m_0} \\ |y_{m_0}(\alpha, K)| < Y_{m_0} \end{pmatrix};$$

par itération on déduit

$$\sum_{m_0}^{\infty} \binom{|x_m(\alpha)|}{|y_m(\alpha)|} \leqslant (|F(\alpha, K)|) \sum_{0}^{\infty} \frac{\left[\left(\int_{m_0\mu}^{\alpha-\mu}(|\mathscr{H}(\alpha, \beta, K)|)(|F(\beta, K)|)\right]^n\right.}{n!} \binom{X_{m_\bullet}}{Y_{m_0}}.$$

Pour avoir le comportement (B-0) désiré, on peut montrer

(B-1)   1) $m_0 = 1$ $(|\mathscr{H}(\alpha, 0,)K|) \leqslant N(K)$,

et $\lim_{\lambda \to \infty} (|\mathscr{H}(\alpha, 0, K)|) \neq 0$ si on veut $\lim_{\lambda \to \infty} (|A(\alpha, K)|) \neq 0$.

(B-2)   2) $\displaystyle\int_{\mu}^{\alpha-\mu} |\mathscr{H}(\alpha, \beta, K)| \, |F(\beta, K)| \mathrm{d}\beta < M(K)$,

a) Equations non couplées $r_c \neq 0$ les matrices sont des éléments 1×1

$$F = B(\alpha, K) \quad \text{et} \quad \mathscr{H} = G_l(\alpha, \beta, K).$$

On peut montrer d'après (9) et (A-4) $|G_l(\alpha, \beta, K)| < N(K)$,

1) $|G_l(\alpha, 0, K)| < N(K)$,

2) $\displaystyle\int_{\mu}^{x-\mu} |B(\beta, K)| \, |G_l(\alpha, \beta, K)| \mathrm{d}\beta < N(K) \int_{\mu}^{\alpha-\mu} |B(\beta, K)| \mathrm{d}\beta$,

b) Equations couplées (20) $r_c \neq 0$; on prend $f_x = f_y = B(\alpha, K)$

$$\mathscr{H}_{J+\eta}^{J\pm\eta} = G_{J+\eta}^{J\pm\eta}.$$

On peut montrer d'après (21) et (A-4)

$$|G_{J+\eta}^{J\pm\eta}(\alpha, \beta, K)| < N_{J+\eta}^{J\pm\eta}(K),$$

1) $|G_{J+\eta}^{J\pm\eta}(\alpha, 0, K)| < N_{J+\eta}^{J\pm\eta}(K)$,

2) $\displaystyle\int_{\mu}^{\alpha-\mu} |B(\beta, K)| \, |G_{J+\eta}^{J\pm\eta}(\alpha, \beta, K)| \mathrm{d}\beta < N_{J+\eta}^{J\pm\eta} \int_{\mu}^{x-\mu} |B(\beta, K)| \mathrm{d}\beta$.

c) Cas $r_c = 0$ on obtient aisément

$$F(x) -\!- \begin{pmatrix} \alpha^{v-1} & 0 \\ 0 & \alpha^{-(\zeta+1)} \end{pmatrix}; \quad (\mathscr{H}) \quad \begin{pmatrix} G_{J-1}^{J-1} \dfrac{1}{\alpha^v(\alpha-2iK)}, & G_{J-1}^{J+1} \dfrac{1}{\alpha^v(\alpha-2iK)} \\ G_{J-1}^{J-1} \dfrac{\alpha^\zeta}{(\alpha-2iK)}, & G_{J+1}^{J+1} \dfrac{\alpha^\zeta}{(\alpha-2iK)} \end{pmatrix},$$

où $G_{J+\eta}^{J\pm\eta}$ est donné par (32).

On établit d'abord 2 relations intermédiares par examen des expressions des noyaux $\mathscr{K}_{J+\eta}^{J\pm\eta}$ pour des potentiels de la forme (30)

$$(B\text{-}3) \qquad |G_{J+\eta}^{J+\eta}(\alpha,\beta,K)| \leqslant A_1(K) \left| \frac{K+i\beta}{K+i\alpha} \right|^{J+\eta} |C_\eta^{\pm\eta}(\alpha-\beta-\mu)| +$$

$$+ A_2(K) \frac{(\alpha-\beta-\mu)^\xi}{|K+i\alpha|} \qquad (\text{avec } \xi = \xi_i \text{ ou } \nu),$$

$$(B\text{-}4) \qquad |G_{J+\eta}^{J-\eta}(\alpha,\beta,K)| \leqslant A_3(K) \frac{|K+i\beta|^{J-\eta}}{|K+i\alpha|^{J+\eta}} |C_T(\alpha-\beta-\mu)| +$$

$$+ A_4(K) \frac{(\alpha-\beta-\mu)^\nu}{|K+i\alpha|^{\iota\eta+1}}.$$

Les majorations (B-3) et (B-4) nous permettent de satisfaire aux conditions (B-1) et (B-2) si $0 < \zeta < 2$ où ζ est le plus petit de $2-\nu$ er $2-\xi_i$. Montrons enfin que ce comportement (31) en α entraine l'existence des limites (29) quand $r \to 0$

$$\lim_{r\to 0} r^{J+1} T(K,r) = \lim_{r\to 0} y_0 r^{J+1} h_{J+1}(Kr) + \sum_{p=0}^{J+1} \lim_{r\to 0} r^{J+1-p} \int_\mu^\infty \exp[-\alpha r]\,\tau(\alpha) \cdot$$

$$\cdot \left(\frac{i}{2(K+i\alpha)} \right)^p \mathrm{d}\alpha\, a_{J+1,p} = \frac{(2J+2)!}{(J+1)!} (-i)^{J+2} \left| \left(\frac{i}{2k} \right)^{J+1} y_0 + \int_\mu^\infty \tau(\alpha) \left(\frac{i}{2(k+i\alpha)} \right)^{J+1} \mathrm{d}\alpha \right|,$$

$$\lim_{r\to 0} r^{J-1+\nu} R(K,r) = (-i)^J \sum_{p=0}^{J-1} \lim_{r\to 0} r^{J-1+\nu-p} \int_\mu^\infty \exp[-\alpha r]\varrho(\alpha) \left(\frac{i}{2(k+i\alpha)} \right)^p a_{J-1,p} \mathrm{d}\alpha,$$

chacun des termes de la somme possède une limite étant donné le comportement de $\varrho(\alpha)$.

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RIASSUNTO (*)

Intendiamo costruire un formalismo che permetta di collegare i parametri fenomenologici dei diversi potenziali nello scattering nucleone-nucleone ai risultati sperimentali: spostamenti di fase e parametri di accoppiamento. Mostro che gli spostamenti di fase si ottengono calcolando numericamente due integrali, i cui integrandi sono soluzioni delle equazioni integrali accoppiate di Volterra, i cui noccioli dipendono esplicitamente dai potenziali.

Il calcolo dell'intensità di diffrazione dei raggi X
da parte di strutture denotanti disordine monodimensionale.

NOTA I. – Criteri matematici.

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(ricevuto il 28 Giugno 1961)

Riassunto. — Nel presente lavoro, che costituisce la parte teorica di una serie di tre note sull'argomento, è fornita una nuova trattazione matematica per il calcolo dell'intensità di diffrazione dei raggi X da parte di strutture monodimensionalmente disordinate. Tale trattazione, che consente in generale una notevole semplicità di calcolo, utilizza il metodo matriciale in una forma sensibilmente diversa da quella già discussa da HENDRICK e TELLER [1]: anzitutto perchè si considera sotto l'aspetto probabilistico la successione dei possibili vettori di traslazione tra strati adiacenti, in luogo della successione di strati in diverse posizioni assolute; quindi perchè è stata evitata la necessità di effettuare qualsivoglia diagonalizzazione di matrici; infine perchè il presente metodo è esteso al caso più generale in cui l'ordine di influenza *s* nella successione probabilistica degli strati, eventualmente di tipo diverso, sia maggiore di 1.

Introduzione.

Il problema del calcolo dell'intensità di diffrazione dei raggi X da parte di strutture denotanti disordine monodimensionale è già stato affrontato in particolare da HENDRICKS e TELLER [1], WILSON [2], JAGODZINSKI [3,4], KAKINOKI e KOMURA [5].

[1] S. B. HENDRICKS and E. TELLER: *Journ. Chem. Phys.*, **10**, 147 (1942).
[2] A. J. C. WILSON: *Proc. Roy. Soc.*, A **180**, 277 (1952).
[3] H. JAGODZINSKI: *Acta Cryst.*, **2**, 201; 208; 298 (1949).
[4] H. JAGODZINSKI: *Acta Cryst.*, **7**, 17 (1954).
[5] J. KAKINOKI and Y. KOMURA: *Journ. Phys. Soc. Japan*, **7**, 30 (1952).

La trattazione di HENDRICKS e TELLER è fondata sul metodo delle matrici di probabilità di sequenza tra strati, mentre la trattazione di WILSON risolve il problema calcolando tali probabilità mediante la risoluzione di un opportuno sistema di equazioni alle differenze finite. JAGODZINSKI, dopo aver esteso ad una maggior generalità il metodo di Wilson, propone una nuova trattazione matematica del problema, fondata sulla risoluzione di sistemi di equazioni dedotti dall'applicazione di opportuni operatori di gruppo ai parametri di probabilità. KAKINOKI e KOMURA sottopongono ad esame critico il metodo delle matrici e quello delle equazioni alle differenze finite, dimostrandone tra l'altro l'equivalenza sostanziale.

Il metodo di risoluzione generale del problema che sarà proposto in questo lavoro si fonda sull'applicazione del metodo matriciale in una forma nuova. Invece di prendere in considerazione, come nelle trattazioni degli autori citati, le probabilità di successione tra strati considerati in posizioni diverse rispetto ad un riferimento fisso, vengono introdotte le probabilità di successione tra i diversi possibili vettori di traslazione tra strati; questo nuovo punto di vista consente, oltre ad una agevole estensione del metodo matriciale a tutti i casi possibili di ordine di influenza tra strati, sia nel caso di strati dello stesso tipo che nel caso di strati di tipo diverso, una maggiore semplicità di calcolo. Il metodo che esporremo non richiede inoltre alcuna diagonalizzazione di matrici [6], a differenza di quello proposto da HENDRICKS e TELLER; nel caso più generale tale diagonalizzazione non è d'altronde sempre effettuabile coi metodi standard [1,6].

Ci proponiamo di illustrare, in questa prima nota, il metodo generale di impostazione matematica del problema; nel primo capitolo verrà studiato il caso di disordine monodimensionale in strutture costituite da strati dello stesso tipo, mentre nel secondo la trattazione verrà estesa al caso in cui siano presenti diversi tipi di strati. Nella seconda nota riporteremo le formule, dedotte sulla base dei formalismi matematici precedentemente discussi, dell'intensità diffratta da parte di alcuni modelli strutturali denotanti tipi diversi di disordine monodimensionale. Nella terza nota infine applicheremo tali formule alla soluzione di alcuni problemi di disordine monodimensionale connessi con le forme a strati, o violette, del tricloruro di titanio, componente del sistema catalitico per la polimerizzazione stereospecifica delle α-olefine [7].

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[6] G. ALLEGRA: *Acta Cryst.*, **14**, 535 (1961).
[7] G. NATTA: Conferenza tenuta al II Congresso Internazionale sulla Catalisi, Parigi, luglio 1960, *Chimica e Industria*, **40**, 1207 (1960).

A) Il problema matematico dell'intensità di diffrazione dei raggi X da strutture con disordine monodimensionale, costituite da strati dello stesso tipo (*).

1. – Introduzione.

Intenderemo per «strato» l'insieme costituito dalla ripetizione infinita per traslazione, secondo due vettori non coincidenti $\boldsymbol{a}$ e $\boldsymbol{b}$, di una medesima unità strutturale; lo strato può essere così propriamente definito come un cristallo bidimensionale.

Ad ogni unità strutturale potrà associarsi un fattore di struttura r, funzione del vettore reciproco $\boldsymbol{S}$ ($\boldsymbol{S} = (2 \sin \theta)/\lambda$), riferito ad una particolare terna di assi solidali con l'unità strutturale stessa. Assumeremo due assi della terna coincidenti con $\boldsymbol{a}$ e $\boldsymbol{b}$ ed il terzo, $\boldsymbol{c}$, perpendicolare ai precedenti (e perpendicolare allo strato). Noi faremo sempre l'ipotesi che ogni strato sia costituito da $N_1 N_2$ celle elementari, con N_1 ed N_2 molto grandi; potremo così definire il fattore di struttura dello strato, V:

$$(1) \qquad V(\boldsymbol{S}) = \begin{cases} N_1 N_2 r(\boldsymbol{S}) \ldots \ldots \text{ se } \begin{vmatrix} \boldsymbol{a} \times \boldsymbol{S} = h \text{ (intero)}, \\ \boldsymbol{b} \times \boldsymbol{S} = k \text{ (intero)} \end{vmatrix} \\ 0 \ldots \ldots \ldots \ldots \text{ negli altri casi}. \end{cases}$$

L'intensità totale diffratta da parte di qualunque struttura costituita da strati sovrapposti sarà in generale esprimibile come somma del termine di interferenza di ogni strato con se stesso, di ogni strato col suo adiacente (nella direzione positiva dell'asse $\boldsymbol{c}$), col suo biadiacente, ecc. ...

Poichè tutti gli strati sono paralleli, il fattore di struttura, rispetto ad un'origine fissa nello spazio, del generico strato di numero indice j, sarà rappresentabile nel modo seguente:

$$(2) \qquad V^{(j)}(\boldsymbol{S}) = V(\boldsymbol{S}) \exp [- i \varphi^{(j)}(\boldsymbol{S})] ;$$

essendo $\varphi^{(j)}(\boldsymbol{S})$ l'angolo di fase corrispondente alla traslazione $\boldsymbol{t}^{(j)}$ dall'origine spaziale fissa all'origine della terna d'assi intrinseca:

$$(3) \qquad \varphi^{(j)}(\boldsymbol{S}) = 2\pi \boldsymbol{t}^{(j)} \times \boldsymbol{S} .$$

(*) Intenderemo per strati dello stesso tipo, qui e nel seguito, strati che si possono ritenere ottenuti gli uni dagli altri per semplice traslazione, nella struttura considerata

Se il numero totale degli strati sovrapposti è N_3, l'intensità totale diffratta, per S qualunque, sarà fornita dall'espressione:

$$(4) \qquad I = N_3 V V^* + V V^* \cdot$$

$$\cdot \left\{ \sum_{q=1}^{N_3-1} \left[\sum_{j=1}^{N_3-q} \exp\left[-i(\varphi^{(j)} - \varphi^{(j+q)})\right] + \sum_{j=1}^{N_3-q} \exp\left[+i(\varphi^{(j)} - \varphi^{(j+q)})\right]\right]\right\};$$

dove $V = V(\boldsymbol{S})$; $\varphi^{(j)} = \varphi^{(j)}(\boldsymbol{S})$.

Indicheremo ora con $\Delta\varphi^{(j,j+q)}$ la differenza $(\varphi^{(j)} - \varphi^{(j+q)})$. Se ammettiamo che tra due strati consecutivi possano aversi più traslazioni statisticamente possibili, sarà fisicamente significativa per noi solo la media dell'espressione (4):

$$(5) \qquad \bar{I} = N_3 V V^* + V V^* \left\{ \sum_{q=1}^{N_3-1} \left[\sum_{j=1}^{N_3-q} \overline{\exp\left[-i\Delta\varphi^{(j,j+q)}\right]} + \sum_{j=1}^{N_3-q} \overline{\exp\left[+i\Delta\varphi^{(j,j+q)}\right]}\right]\right\}.$$

Data l'equivalenza tra gli strati, potremo scrivere

$$(6) \qquad \overline{\exp\left[\mp i\Delta\varphi^{(j,j+q)}\right]} =$$

$$= \overline{\exp\left[\mp i\Delta\varphi^{(j,j+1)}\right] \cdot \exp\left[\mp i\Delta\varphi^{(j+1,j+2)}\right] \ldots \exp\left[\mp i\Delta\varphi^{(j+q-1,j+q)}\right]} = \overline{\exp\left[\mp i\Delta\varphi^{(q)}\right]};$$

ed elimineremo così dalla (5) le sommatorie rispetto all'indice j:

$$(5') \qquad \bar{I} = V V^* \left\{ N_3 + \sum_{q=1}^{N_3-1} (N_3 - q)\left(\overline{\exp\left[-i\Delta\varphi^{(q)}\right]} + \overline{\exp\left[+i\Delta\varphi^{(q)}\right]}\right)\right\}.$$

Passando quindi specificamente a trattare del caso di successioni statisticamente disordinate degli strati, considereremo separatamente i casi in cui l'ordine di influenza (s) nella successione probabilistica dei medesimi sia 1, o 2, o maggiore di 2.

2. – Trattazione del caso con $s = 1$.

Nel caso semplice in cui la sfera di influenza probabilistica si estenda solo tra uno strato e l'adiacente, il calcolo dei valori medi $\overline{\exp\left[\mp i\Delta\varphi^{(q)}\right]}$, secondo la (6), è particolarmente semplice. Infatti in tal caso ogni termine del tipo $\exp\left[\mp i\Delta\varphi^{(k,k+1)}\right]$ è indipendente dagli altri; in queste condizioni, la media del prodotto rappresentato nel 2° membro dell'eguaglianza (6) è data dal prodotto delle medie dei singoli termini:

$$(6') \qquad \overline{\exp\left[\mp i\Delta\varphi^{(q)}\right]} = \left(\overline{\exp\left[\mp i\Delta\varphi^{(1)}\right]}\right)^q.$$

Siano n le possibili traslazioni mediante le quali ogni strato succede al precedente, e sia t_h ($h = 1, 2, ..., n$) il vettore che rappresenta la generica traslazione. Se φ_h è la fase corrispondente alla traslazione t_h, si avrà

$$(7) \qquad \overline{\exp{[\mp i\Delta\varphi^{(1)}]}} = \sum_{h=1}^{n} p_h \exp{[\mp i\varphi_h]},$$

essendo p_h la probabilità che si verifichi la n-esima traslazione. Dalla (7) si deduce subito in particolare che $|\overline{\exp{[\mp i\Delta\varphi^{(1)}]}}| \leqslant 1$. La (5') può ora essere scritta:

$$(8) \quad \bar{I} = VV^* \left\{ -N_3 + \sum_{q=0}^{N_3-1} (N_3 - q) \left[(\overline{\exp{[-i\Delta\varphi^{(1)}]}})^q + (\overline{\exp{[+i\Delta\varphi^{(1)}]}})^q \right] \right\}.$$

Ricordando ora che, per $|a| < 1$:

$$(9) \qquad S(a) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} [(N-i)a^i] = (1-a)^{-1}:$$

la (8) diviene, per N_3 molto grande e per tutti i valori di S per cui $|\overline{\exp{[\mp i\Delta\varphi^{(1)}]}}| < 1$:

$$(8') \qquad \bar{I} = N_3 VV^* \left\{ -1 + \frac{1}{1 - \overline{\exp{[-i\Delta\varphi^{(1)}]}}} + \frac{1}{1 - \overline{\exp{[+i\Delta\varphi^{(1)}]}}} \right\};$$

ovvero, sviluppando i calcoli ed indicando con I_{Av} l'intensità media diffratta da ogni strato:

$$(8'') \qquad I_{Av} = \frac{\bar{I}}{N_3} = VV^* \frac{1 - \overline{\exp{[-i\Delta\varphi^{(1)}]}} \cdot \overline{\exp{[+i\Delta\varphi^{(1)}]}}}{1 + \overline{\exp{[-i\Delta\varphi^{(1)}]}} \cdot \overline{\exp{[+i\Delta\varphi^{(1)}]}} - 2\,\mathrm{Re}\,(\overline{\exp{[i\Delta\varphi^{(1)}]}})},$$

($\mathrm{Re} = $ « parte reale di »). Per i valori di S per cui sia $\overline{\exp{[\mp i\Delta\varphi^{(1)}]}} = 1$, dalla (8) si ottiene invece la nota formula dell'interferenza classica di Bragg. Se infatti scriviamo: $\exp{[\mp i\Delta\varphi^{(1)}]} = \exp{[\mp i\alpha]}$, la (8) diviene:

$$(8''') \qquad \bar{I} = VV^* \frac{\sin^2 \dfrac{(N_3 - 1)\alpha}{2}}{\sin^2 \dfrac{\alpha}{2}}.$$

Qualora invece si avesse

$$|\overline{\exp{[\mp i\Delta\varphi^{(1)}]}}| = 1,$$

per *ogni* valore di $\boldsymbol{S}$, allora l'intensità diffratta sarebbe sempre espressa dalla (8''')
e la struttura sarebbe tridimensionalmente ordinata. Infatti in tal caso (vedi
la (7)) dovrebbe aversi: $p_h = 1$, $p_k(h \neq k) = 0$; oppure, che è fisicamente la
stessa cosa, $\varphi_h \equiv \varphi_k$ (h e k qualunque), e quindi $\boldsymbol{t}_h = \boldsymbol{t}_k$.

3. – Trattazione del caso con $s = 2$.

Quando la posizione relativa di uno strato è influenzata probabilisticamente
dalla posizione relativa dei due strati che lo precedono, la media rappresen-
tata in (6) non può valutarsi nel modo semplice ora descritto. In questo caso
infatti ogni termine di frangia $\exp[\mp i \Delta \varphi^{(k,k+1)}]$ è influenzato dal termine di
frangia precedente $\exp[\mp i \Delta \varphi^{k-1,k}]$.

Indicheremo nel seguito con f_h la frequenza di comparizione della trasla-
zione $\boldsymbol{t}_h$, e con p_{hk} la probabilità che alla traslazione $\boldsymbol{t}_h$ succeda la $\boldsymbol{t}_k$; le gran-
dezze f e p sono collegate dal sistema di n equazioni:

$$(10) \qquad\qquad f_k = \sum_{h=1}^{n} f_h p_{hk} \qquad\qquad (k = 1, 2, ..., n);$$

dal quale, note ad esempio le probabilità di successione p_{hk}, possono immedia-
tamente ricavarsi le frequenze f_k.

Indicando quindi ancora con φ_h i valori degli angoli di fase corrispondenti
alle traslazioni $\boldsymbol{t}_h$, la media indicata nell'espressione (6) si trasforma nel caso
nostro nel modo seguente:

$$(11) \quad \overline{\exp[\mp i \Delta \varphi^{(q)}]} = \sum_h \sum_k \sum_l ... f_h \exp[\mp i \varphi_h] \cdot p_{hk} \exp[\mp i \varphi_k] \cdot p_{kl} \exp[\mp i \varphi_l] ...;$$

dove in ognuno dei termini della sommatoria sono inclusi q fattori del tipo
$\exp[\mp i \varphi_s]$.

Definendo ora un vettore riga così costituito:

$$(12) \qquad\qquad \mathscr{F} = |f_1 \exp[-i\varphi_1] \qquad f_2 \exp[-i\varphi_2] ... f_n \exp[-i\varphi_n]|;$$

una matrice quadrata $\boldsymbol{Q}$ così definita:

$$(13) \qquad\qquad \boldsymbol{Q} = |Q_{hk}| = |p_{hk} \exp[-i\varphi_k]|;$$

e un vettore colonna $\boldsymbol{1}$ costituito da n termini unitari, l'espressione (10) può
essere così sinteticamente rappresentata;

$$(14) \qquad \overline{\exp[-i\Delta\varphi^{(q)}]} = \mathscr{F} \boldsymbol{Q}^{q-1} \boldsymbol{1}; \qquad \overline{(\exp[+i\Delta\varphi^{(q)}]} = \mathscr{F}^*(\boldsymbol{Q}^*)^{q-1}\boldsymbol{1});$$

mentre l'espressione (5′) si trasforma, in questo caso, nella

$$(15) \qquad \bar{I} = VV^* \Big\{ N_3 + \sum_{q=1}^{N_3-1} (N_3 - q)(\boldsymbol{\mathscr{F}} \boldsymbol{Q}^{q-1} \mathbf{1} + \boldsymbol{\mathscr{F}}^* (\boldsymbol{Q}^*)^{q-1} \mathbf{1}) \Big\} =$$

$$= VV^* \Big\{ N_3 + \boldsymbol{\mathscr{F}} \Big(\sum_{q=1}^{N_3-1} (N_3 - q) \boldsymbol{Q}^{q-1} \Big) \mathbf{1} + \boldsymbol{\mathscr{F}}^* \Big(\sum_{q=1}^{N_3-1} (N_3 - q)(\boldsymbol{Q}^*)^{q-1} \Big) \mathbf{1} \Big\}.$$

Ogni termine della matrice $\boldsymbol{Q}^q$, quando q tende all'infinito, tende in generale a 0, in una struttura statistica.

Infatti dalla definizione (13) discende:

$$(16) \qquad |(Q^q)_{hk}| = \Big| \sum_{l_1} \sum_{l_2} \dots \sum_{l_{q-1}} Q_{h l_1} Q_{l_1 l_2} \dots Q_{l_{q-1} k} \Big| =$$
$$\scriptstyle (l_1, l_2 \dots l_{q-1}=1,\dots n)$$

$$= \Big| \sum_{l_1} \dots \sum_{l_{q-1}} p_{h l_1} \exp\left[-i\varphi_{l_1} \right] \dots p_{l_{q-1} k} \exp\left[-i\varphi_k \right] \Big| \leqslant \sum_{l_1} \dots \sum_{l_{q-1}} p_{h l_1} p_{l_1 l_2} \dots p_{l_{q-1} k} = p_{hk}^{(q)}.$$

Ma $p_{hk}^{(q)}$ rappresenta la probabilità, minore o al più eguale a uno per definizione, che alla traslazione $\boldsymbol{t}_h$ succeda, dopo q traslazioni, la traslazione $\boldsymbol{t}_k$. Questo significa anzitutto che

$$|(Q^q)_{hk}| \leqslant 1 ,$$

qualunque sia q.

Ma in una struttura statistica i termini $\exp\left[-i\varphi_l \right]$ (vedi espressione (16)) non sono in generale in fase tra loro; di conseguenza in generale il modulo di $(Q^q)_{hk}$ diviene sempre più piccolo, al crescere di q, rispetto a $p_{hk}^{(q)}$. Ciò esprime la convergenza a zero, al limite di $q \to \infty$, di $(Q^q)_{hk}$. In queste condizioni si verifica inoltre la convergenza della serie matriciale:

$$\boldsymbol{\mathscr{Q}} = \sum_{q=0}^{\infty} \boldsymbol{Q}^q = (\boldsymbol{E} - \boldsymbol{Q})^{-1} ;$$

ed inoltre risulta convergente allo stesso limite la serie:

$$(17) \qquad S(\boldsymbol{Q}) = \lim_{N_3 \to \infty} \frac{1}{N_3} \sum_{q=1}^{N_3-1} (N_3 - q) \boldsymbol{Q}^{q-1} = \boldsymbol{\mathscr{Q}} = (\boldsymbol{E} - Q)^{-1} ;$$

in tutta analogia con la serie numerica (9).

L'espressione (15) si trasforma quindi, per N_3 molto grande, nella

$$(15') \qquad I_{\text{Av}} = \frac{\bar{I}}{N_3} = VV^* \{ 1 + 2 \operatorname{Re}\left[\boldsymbol{\mathscr{F}} (\boldsymbol{E} - \boldsymbol{Q})^{-1} \mathbf{1} \right] \}.$$

Tuttavia in una struttura statistica vi possono essere, come nel caso di $s = 1$, dei valori di S tali per cui la serie (17) risulti divergente. Tali valori si possono ottenere nel modo seguente, in analogia al caso semplice rappresentato dalla serie numerica (9). Quest'ultima diverge quando si abbia $a - 1$, nel qual caso l'espressione all'ultimo membro della (9) diventa indeterminata. Il generico elemento della matrice $(\boldsymbol{E} - \boldsymbol{Q})^{-1}$ (o della $(\boldsymbol{E} - \boldsymbol{Q}^*)^{-1}$) è uguale al rapporto tra il minore complementare del corrispondente elemento della matrice e il determinante della matrice stessa. Affinchè l'espressione (17) divenga indeterminata, sarà quindi sufficiente che il determinante della matrice $(\boldsymbol{E} - \boldsymbol{Q})$ si annulli: questa è infatti la condizione di indeterminazione per ogni elemento della matrice $(\boldsymbol{E} - \boldsymbol{Q})^{-1}$. Diremo pertanto che la serie (17) diverge per tutti i valori di S per cui si abbia;

$$(18) \qquad \mathrm{Det}\,(\boldsymbol{E} - \boldsymbol{Q}) = \mathrm{Det}\,(\boldsymbol{E} - \boldsymbol{Q}^*) = 0 \,.$$

In tutti questi punti dello spazio reciproco il calcolo dell'intensità media diffratta non può più effettuarsi secondo la (15); tale calcolo va allora effettuato caso per caso, ottenendosi in generale i picchi caratteristici delle riflessioni di Bragg. Vedremo nella II nota esempi concreti di verifica di questo evento, in alcuni tipi di strutture statistiche.

4. – Trattazione del caso con $s > 2$.

La trattazione matematica fin qui esposta può ancora estendersi al caso più complesso in cui la posizione relativa di uno strato è probabilisticamente influenzata dalle posizioni relative degli s strati che lo precedono ($s > 2$) oppure, che è lo stesso, la traslazione tra due strati consecutivi è influenzata dalle precedenti ($s - 1$) traslazioni.

Applicheremo nel seguito alle sequenze di vettori di traslazione la rappresentazione per « complessioni » già introdotta da JAGODZINSKI ([4]) per le sequenze di strati occupati definite posizioni. Indicheremo con l_1, l_2, ..., l_j, ..., l_q i numeri indice dei vettori di traslazione $\boldsymbol{t}_{l_j}$ che hanno luogo rispettivamente tra primo e secondo, secondo e terzo ... q-esimo e ($q+1$)-esimo strato, a partire da un vettore di riferimento, arbitrariamente scelto: l'indice l_j, con j qualunque, è variabile tra 1 ed n, se n sono le traslazioni possibili tra uno strato e il successivo.

Indicheremo nel seguito con $u_{(q)}$ il numero indice, variabile da 1 a n^q, delle possibili complessioni di q traslazioni successive, nel caso che sia $q \leqslant s - 1$:

$$(19) \qquad u_{(q)} \equiv (l_1 l_2 \dots l_q) \,.$$

Omettendo per semplicità di indicare l'indice suffisso q quando sia $q = s - 1$, indicheremo con f_u $(u = (l_1 l_2 \ldots l_{s-1}))$ la frequenza di comparizione della complessione di $s - 1$ traslazioni successive caratterizzata dall'indice u. Sia ora $p_{u l_s}$ la probabilità che a tale complessione succeda la traslazione l_s; la complessione di ordine s: $u l_s = (l_1 l_2 \ldots l_{s-1} l_s)$ potrà essere caratterizzata mediante la successione delle due complessioni di ordine $(s - 1)$: $u \equiv (l_1 l_2 \ldots l_{s-1})$; $u' \equiv (l_2 l_3 \ldots l_s)$, e potremo quindi associare indici omogenei alla probabilità ora definita:

$$(20) \qquad p_{u l_s} \equiv p_{u u'} .$$

Indicheremo inoltre con φ_u ($\varphi_{u_{(q)}}$ per $q < s - 1$) la somma degli $s - 1$ angoli di fase corrispondenti alle traslazioni che costituiscono la complessione di indice u:

$$(21) \qquad \varphi_u = \varphi_{l_1 l_2 \ldots l_{s-1}} = \varphi_{l_1} + \varphi_{l_2} + \ldots + \varphi_{l_{s-1}} .$$

Le grandezze f e p sono legate tra loro da relazioni analoghe alle (10):

$$(22) \qquad f_{u'} = \sum_{u=1}^{n^{s-1}} f_u p_{u u'} .$$

Se le probabilità p sono note, o comunque dedotte, le relazioni (22) costituiscono un sistema lineare omogeneo di n^{s-1} equazioni in altrettante incognite f. Si può facilmente dimostrare che il determinante dei coefficienti delle incognite di tale sistema è nullo necessariamente, se i valori p sono correttamente assegnati: quindi, date le probabilità p di successione, le frequenze di comparizione f sono dedotte in modo univoco. Valuteremo quindi nel modo seguente la media indicata in (6), distinguendo anzitutto tra i casi in cui si abbia $q \leqslant s - 1$ e $q > s - 1$.

Nel caso in cui sia $q \leqslant s - 1$ si avrà, evidentemente,

$$(23) \qquad \exp\left[- i \Delta \varphi^{(q)}\right] = \sum_{u_{(q)}=1}^{n^q} f_{\cdot(q)} \exp\left[- i \varphi_{\cdot(q)}\right] .$$

Le frequenze $f_{u_{(q)}}$ $(q < s - 1)$ si possono facilmente calcolare in funzione delle frequenze f_u, supposte note, dalle relazioni:

$$(24) \qquad f_{u_{(q)}} = f_{l_1 l_2 \ldots l_q} = \sum_{l_{q+1}} \ldots \sum_{l_{s-1}} f_{l_1 l_2 \ldots l_q l_{q+1} \ldots l_{s-1}} .$$

Nel caso in cui sia $q \geqslant s - 1$, la media (6) può essere invece così rappre-

sentata :

$$(23') \quad \overline{\exp\left[-i\Delta\varphi^{(q)}\right]} = \sum_{l_1} \dots \sum_{l_q} f_{l_1\dots l_{s-1}} \exp\left[-i(\varphi_{l_1}+\varphi_{l_2}+\dots+\varphi_{l_{s-1}})\right]p_{l_1 l_2 \dots l_s} \cdot$$

$$\cdot \exp\left[-i\varphi_{l_s}\right]\cdot p_{l_2 l_3 \dots l_{s+1}} \exp\left[-i\varphi_{l_{s+1}}\right] \dots p_{l_{q-s+1} \dots l_q} \cdot \exp\left[-i\varphi_{l_q}\right] =$$

$$= \sum_{u_{(1)}} \sum_{u_{(2)}} \dots \sum_{u_{(q-s+2)}} f_{u_{(1)}} \exp\left[-i\varphi_{u_{(1)}}\right] Q_{u_{(1)}u_{(2)}} \dots Q_{u_{(q-s+1)}u_{(q-s+2)}} \cdot$$

avendo indicato con $Q_{uu'}$ le grandezze :

$$(24) \qquad Q_{uu'} = p_{uu'} \exp\left[-i\varphi_{l_s}\right]\left(u \equiv (l_1 l_2 \dots l_{s-1}); \ u' \equiv (l_2 \dots l_s)\right) .$$

Introducendo ora, in analogia al caso trattato nella sezione precedente, il vettore riga di ordine n^{s-1}:

$$(25) \qquad \mathscr{F}' = |f_u \exp\left[-i\varphi_u\right]|; \qquad\qquad u = 1, 2, \dots, n^{s-1};$$

la matrice quadrata $\boldsymbol{Q}'$, pure di ordine n^{s-1}:

$$(26) \qquad \boldsymbol{Q}' = |Q_{uu'}|$$

e il vettore colonna $\mathbf{1}$, dello stesso ordine, la (23') può scriversi sinteticamente:

$$(27) \qquad \overline{\exp\left[-i\Delta\varphi^{(q)}\right]} = \mathscr{F}'\boldsymbol{Q}'^{(q-s+1)}\mathbf{1};$$

e finalmente l'espressione dell'intensità media diffratta (5') diviene:

$$(28) \qquad \bar{I} = VV^*\left\{N_3 + 2\sum_{q=1}^{s-2}(N_3-q)\left(\sum_{u_{(q)}=1}^{n^q} f_{v_{(q)}}\cos\varphi_{v_{(q)}}\right) + \right.$$

$$\left. + 2\operatorname{Re}\left[\sum_{q=s-1}^{N_3-1}(N_3-q)\,\mathscr{F}'\boldsymbol{Q}'^{(q-s+1)}\mathbf{1}\right]\right\} .$$

Svolgendo quindi considerazioni analoghe a quelle già indicate nella sezione precedente, la (28) si trasforma nell'espressione seguente dell'intensità media diffratta per strato elementare, valida per s qualunque $(N_3 \to \infty)$:

$$(28') \qquad I_{\mathrm{Av}} = \frac{\bar{I}}{N_3} = VV^*\left\{1 + 2\sum_{q=1}^{s-2}\left(\sum_{u_{(q}=1}^{n^q} f_{u_{(q)}}\cos\varphi_{v_{(q)}}\right) + 2\operatorname{Re}\left[\mathscr{F}'(\boldsymbol{E}-\boldsymbol{Q}')\mathbf{1}\right]\right\} .$$

Il termine di più laboriosa valutazione nella espressione (28') è di regola il termine $\mathscr{F}'(\boldsymbol{E}-\boldsymbol{Q}')^{-1}\mathbf{1}$, data la necessità di calcolare la matrice inversa $(\boldsymbol{E}-\boldsymbol{Q}')^{-1}$. Si fa tuttavia osservare che, nonostante l'ordine della matrice $\boldsymbol{Q}'$

sia n^{s-1}, il numero di termini non nulli in essa contenuti, anziché essere n^{2s-2}, è al massimo n^s; questo fatto riduce molto la complessità dei calcoli. Infatti ogni termine di probabilità $p_{uu'}$, per avere significato fisico — e quindi essere non nullo — deve soddisfare alla ovvia condizione che la coppia di indici (uu') corrisponda ad una complessione di s traslazioni consecutive $(l_1 \ldots l_s)$. Ma tali complessioni distinte sono in numero n^s; tanti sono perciò i termini non nulli della matrice $\boldsymbol{Q}'$.

Le condizioni di divergenza, al tendere di N_3 all'infinito, della sommatoria rappresentata dall'ultimo termine della (28), sono in tutto analoghe a quelle indicate a proposito della (15), e si esprimono ancora matematicamente colla condizione:

$$(29) \qquad \mathrm{Det}\,(\boldsymbol{E} - \boldsymbol{Q}') = 0 ;$$

in corrispondenza ai valori di $\boldsymbol{S}$ che soddisfano a tale condizione la radiazione diffratta ha in generale lo stesso andamento a picchi infinitamente concentrati caratteristico di qualunque struttura ordinata.

B) Il problema matematico dell'intensità dei raggi X diffratti da strutture con disordine monodimensionale, costituite da strati di tipo diverso.

1. – Trattazione del caso con $s = 1$. I possibili vettori di traslazione tra strati adiacenti non siano univocamente determinati dalla natura degli strati stessi.

Il problema trattato in questa sezione amplia la trattazione di HENDRICKS e TELLER [1] in quanto considera la possibilità che due determinati tipi di strati non si succedano secondo un'unica traslazione, ma secondo più possibili traslazioni, con differenti probabilità.

Siano $1, 2, \ldots, j, \ldots, m$ i numeri indice degli m diversi tipi di strati che si succedono statisticamente nella struttura considerata: allo j-esimo tipo di strato corrisponde un fattore di struttura, riferito ad un'origine solidale con lo strato stesso, $V^{(j)}$. Siano inoltre $\boldsymbol{t}_1^{(ij)}, \boldsymbol{t}_2^{(ij)}, \ldots, \boldsymbol{t}_h^{(ij)}, \ldots, \boldsymbol{t}_{n_{ij}}^{(ij)}$ le n_{ij} diverse possibili traslazioni secondo cui uno strato di tipo j può succedere ad uno strato di tipo i. Corrispondentemente definiremo le probabilità $p_h^{(ij)}$ $(h = 1, \ldots, n_{ij})$ che allo strato di tipo i segua lo strato di tipo j con la traslazione di indice h.

Definiremo quindi le funzioni di frangia medie associate alle traslazioni ora definite, analoghe alla $\exp[+i\,\Delta\varphi^{(1)}]$ della (A-6) ma limitate alle traslazioni tra strati di tipo determinato: $\exp[\mp i\,\overline{\Delta\varphi_{ij}^{(1)}}]$. Evidentemente si ha

$$(1) \qquad \overline{\exp[\mp i\Delta\varphi_{ij}^{(1)}]} = \sum_{n=1}^{n_{ij}} p_h^{(ij)} \exp[\mp\varphi_h^{(ij)}]; \qquad \varphi_h^{(ij)} = 2\pi\boldsymbol{t}_h^{(ij)} \times \boldsymbol{S} .$$

Un sistema omogeneo di equazioni lineari collega le frequenze di comparizione $f^{(i)}$ degli strati di tipo generico i con le probabilità $p_h^{(ij)}$. Tali equazioni assumono la forma seguente:

$$(2) \qquad f^{(i)} = \sum_{j=1}^{m} f^{(j)} \sum_{h=1}^{n_{ji}} p_h^{(ji)} \; ;$$

e sono evidentemente tante quanti sono i tipi di strati.

Passando quindi al calcolo dell'intensità media totale diffratta, valuteremo separatamente, come nei calcoli effettuati nel capitolo precedente, i contributi medi alla diffrazione di ogni strato con se stesso, coll'adiacente, col biadiacente, ecc.

Il contributo medio totale alla diffrazione di ogni strato con se stesso, essendo N_3 il numero totale di strati, vale:

$$(3) \qquad \bar{I}_{(0)} = N_3 \sum_{j=1}^{m} f^{(j)} V^{(j)} V^{(j)*} \; .$$

Risulta utile esprimere tale contributo in una nuova forma. Siano:

$\boldsymbol{V}$ il vettore riga costituito dagli elementi: $V^{(1)} V^{(2)} \ldots V^{(m)}$;

$\widetilde{\boldsymbol{V}}$ il vettore colonna corrispondente;

$\boldsymbol{F}$ la matrice diagonale di ordine m: $F^{(ij)} = f^{(i)} \delta^{(ij)}$.

L'ordine di questi vettori e matrici eguaglia evidentemente il numero totale dei tipi diversi di strati. L'espressione (3) si può scrivere allora:

$$(3') \qquad \bar{I}_{(0)} = N_3 \boldsymbol{V} \boldsymbol{F} \widetilde{\boldsymbol{V}}^* \; .$$

Il contributo medio totale alla diffrazione di ogni strato coll'adiacente vale:

$$(4) \qquad \bar{I}_{(1)} = 2(N_3 - 1) \, \mathrm{Re} \Big\{ \sum_{i=1}^{m} f^{(i)} V^{(i)} \Big[\sum_{j=1}^{m} \overline{\exp \left[- i \, \Delta \varphi_{ij}^{(1)} \right]} \Big] V^{(j)*} \Big\};$$

tale espressione può essere pure rappresentata in forma sintetica introducendo la nuova matrice quadrata $\boldsymbol{Q}$ dello stesso ordine di $\boldsymbol{F}$:

$$(5) \qquad |Q^{(ij)}| = \overline{\left| \exp \left[- i \, \Delta \varphi_{ij}^{(1)} \right] \right|} \; .$$

La (4) diviene quindi:

$$(4') \qquad \bar{I}_{(1)} = 2(N_3 - 1) \, \mathrm{Re} \, \{ \boldsymbol{V} \boldsymbol{F} \boldsymbol{Q} \widetilde{\boldsymbol{V}}^* \} \; .$$

Passando ora a calcolare il contributo medio alla diffrazione tra strati bi-adiacenti, otterremo un'espressione identica alla (4') in cui, in luogo delle medie $\overline{\exp\left[-i\Delta q_{ij}^{(1)}\right]}$ dovremo introdurre le funzioni medie di frangie $\overline{\exp\left[-i\Delta q_{ij}^{(2)}\right]}$, e, in luogo del coefficiente (N_3-1), un coefficiente (N_3-2). Se indichiamo con $\boldsymbol{Q}'$ una matrice analoga alla $\boldsymbol{Q}$ in cui i termini $\exp\left[-i\Delta q_{ij}^{(1)}\right]$, siano ordinatamente sostituiti dai termini $\overline{\exp\left[-i\Delta\varphi_{ij}^{(2)}\right]}$ potremo quindi scrivere:

$$(6) \qquad \bar{I}_{(2)} = 2(N_3-2)\,\mathrm{Re}\,\{\boldsymbol{V}\boldsymbol{F}\boldsymbol{Q}'\widetilde{\boldsymbol{V}}^*\}\,.$$

Definiremo ora la relazione che intercorre tra le medie $\overline{\exp\left[-i\Delta q_{ij}^{(2)}\right]}$ e le corrispondenti $\overline{\exp\left[-i\Delta q_{ij}^{(1)}\right]}$. Poichè, ad esempio, ad uno strato di tipo 1 può succedere, dopo due strati, uno strato dello stesso tipo secondo le possibili successioni:

$$(111),\ (121),\ (131),\ \ldots,$$

si deduce la seguente relazione:

$$(7) \qquad \overline{\exp\left[-i\Delta\varphi_{ij}^{(2)}\right]} = \sum_{k=1}^{m} \overline{\exp\left[-i\Delta\varphi_{ik}^{(1)}\right]}\,\overline{\exp\left[-i\Delta\varphi_{kj}^{(1)}\right]}\,.$$

Ne deriva immediatamente, per la regola del prodotto di matrici, che la matrice $\boldsymbol{Q}'$ è rappresentata dal prodotto della matrice $\boldsymbol{Q}$ per se stessa:

$$(8) \qquad \boldsymbol{Q}' = \boldsymbol{Q}\times\boldsymbol{Q} = \boldsymbol{Q}^2\,.$$

Per induzione generale potremo quindi scrivere:

$$(9) \qquad \bar{I}_{(n)} \doteq 2(N_3-n)\,\mathrm{Re}\,\{\boldsymbol{V}\boldsymbol{F}\boldsymbol{Q}^n\widetilde{\boldsymbol{V}}^*\}\,,$$

e l'espressione dell'intensità totale diffratta diviene:

$$(10) \qquad \bar{I} = \sum_{n=0}^{N_3-1} \bar{I}_{(n)} = N_3\boldsymbol{V}\boldsymbol{F}\widetilde{\boldsymbol{V}}^* + 2\sum_{n=1}^{N_3-1}(N_3-n)\,\mathrm{Re}\,\{\boldsymbol{V}\boldsymbol{F}\boldsymbol{Q}^n\widetilde{\boldsymbol{V}}^*\} =$$

$$= 2\,\mathrm{Re}\,\left\{\boldsymbol{V}\boldsymbol{F}\left[\sum_{n=0}^{N_3-1}(N_3-n)\,\boldsymbol{Q}^n\right]\widetilde{\boldsymbol{V}}^*\right\} - N_3\boldsymbol{V}\boldsymbol{F}\boldsymbol{V}^*\,.$$

In tutta analogia con l'espressione $(A\text{-}15)$, l'espressione (10) risulterà convergente, per N_3 tendente all'infinito, quando sia soddisfatta la relazione:

$$(11) \qquad \mathrm{Det}\,(\boldsymbol{E}-\boldsymbol{Q}) \neq 0\,;$$

ottenendosi il seguente valore per I_{Av}:

$$(12) \qquad I_{\mathrm{Av}} = \frac{\bar{I}}{N_3} = 2 \, \mathrm{Re} \left\{ \boldsymbol{V} \boldsymbol{F} (\boldsymbol{E} - \boldsymbol{Q})^{-1} \widetilde{\boldsymbol{V}}^* \right\} - \sum_{j=i}^{m} f^{(j)} V^{(j)} V^{(j)*} .$$

La trattazione di HENDRICKS e TELLER, considera, come già detto, che tra due strati di tipo determinato possa aver luogo una sola traslazione; questo imporrebbe la necessità di considerare di tipo diverso tutti gli strati che possono succedere ad uno strato generico purchè differenti siano i relativi vettori di traslazione. Vedremo nella nota successiva un esempio concreto che dimostra la notevole portata delle semplificazioni introdotte ([6]).

2. – Trattazione del caso con $s > 1$.

Lo sviluppo di quest'ultima, più generale sezione discenderà in modo diretto dall'applicazione dei concetti già esposti nelle sezioni $(A\text{-}4)$ e $(B\text{-}1)$ di questa nota.

Al fine di non complicare troppo il simbolismo, limiteremo il nostro studio al caso in cui siano presenti due soli tipi di strati, α e β; indicheremo con V^α e V^β i corrispondenti fattori di struttura.

Distingueremo quattro diverse classi di vettori di traslazione: $(\alpha\alpha)$, $(\alpha\beta)$, $(\beta\alpha)$, $(\beta\beta)$, a seconda dei diversi tipi di strati da essi collegati. Sia $n_{(\alpha\alpha)}$, $n_{(\alpha\beta)}$, $n_{(\beta\alpha)}$ e $n_{(\beta\beta)}$ il numero totale corrispondente di possibili traslazioni: sia inoltre $n = n_{(\alpha\alpha)} + n_{(\beta\alpha)} + n_{(\beta\alpha)} + n_{(\beta\beta)}$.

Introdurremo anche in questo caso la rappresentazione di complessioni di q traslazioni consecutive con $q \leqslant s - 1$, mediante l'indice $u_{(q)}$, variabile tra 1 e n^q. Attribuiremo poi all'indice $u_{(q)}$ volta a volta gli ulteriori indici $(\alpha\alpha)$, $(\alpha\beta)$, $(\beta\alpha)$, $(\beta\beta)$, distinguendo così l'insieme delle complessioni di ordine q nelle quattro classi corrispondenti, a seconda che la somma dei vettori di traslazione che compongono la complessione considerata colleghi strati di tipo indicato. L'indice $u_{(q)}$ assumerà valori sempre crescenti nell'ordine tra le quattro classi rappresentate; sarà cioè:

$$u_{(q)}(\alpha\alpha) < u_{(q)}(\alpha\beta) < u_{(q)}(\beta\alpha) < u_{(q)}(\beta\beta) .$$

Il numero di complessioni contenute entro ciascuna classe sarà rispettivamente $n_{(q)}(\alpha\alpha)$, $n_{(q)}(\alpha\beta)$, $n_{(q)}(\beta\alpha)$, $n_{(q)}(\beta\beta)$. La somma di questi 4 numeri eguaglierà naturalmente n^q. Scriveremo nel seguito per semplicità l'indice u senza il suffisso (q) quando si abbia $q = s - 1$.

L'angolo di fase corrispondente alla somma delle traslazioni rappresentate dalla complessione $(u_{(q)})$ potrà ancora definirsi in tutta analogia con quanto posto

nel precedente capitolo:

$$(13) \qquad q_{u_{(q)}} = \varphi_{l_1} + \varphi_{l_2} + \dots + \varphi_{l_q} ; \qquad\qquad u_{(q)} \equiv (l_1 l_2 \dots l_q) ,$$

e così pure le grandezze $p_{uu'}$ e $Q_{uu'}$:

$$(14) \qquad Q_{uu'} = Q_{l_1 l_2 \dots l_s} = p_{l_1 l_2 \dots l_s} \exp\left[- i\varphi_{l_s}\right] .$$

Un sistema di equazioni, analogo al sistema (A-19), collegherà anche in questo caso le frequenze di comparizione f_u con le probabilità di successione $p_{uu'}$. Le frequenze di comparizione delle complessioni di ordine q $(< s-1)$ potranno essere ricavate da equazioni dello stesso tipo delle (A-22). Potranno inoltre in modo analogo essere ricavate le frequenze di comparizione f^α ed f^β degli strati stessi:

$$(15) \qquad f^\alpha = \sum_{u(\alpha\alpha)} f_{u(\alpha\alpha)} + \sum_{u(\alpha\beta)} f_{u(\alpha\beta)} , \qquad f^\beta = \sum_{u(\beta\alpha)} f_{u(\beta\alpha)} + \sum_{u(\beta\beta)} f_{u(\beta\beta)} .$$

Ciò premesso, in analogia concettuale con quanto sviluppato nel capitolo e nella sezione precedenti, potremo effettuare il calcolo dei contributi medi totali alla diffrazione di ogni strato con se stesso ($\bar{I}_0$) coll'adiacente ($\bar{I}_1$), col biadiacente ($\bar{I}_2$)..., col q-adiacente ($\bar{I}_q$), ecc.:

$$(16) \qquad \bar{I}_0 = N_3\{f^\alpha V^\alpha V^{\alpha*} + f^\beta V^\beta V^{\beta*}\} ;$$

$$(17) \qquad \bar{I}_q(q < s-1) = (N_3 - q)\Big\{ V^\alpha V^{\alpha*}\Big[2\sum_{u_q(\alpha\alpha)} f_{u_q(\alpha\alpha)} \cos\varphi_{u_q(\alpha\alpha)}\Big] +$$

$$+ (V^\alpha V^{\beta*} + V^\beta V^{\alpha*})\Big[2\sum_{u_q(\alpha\beta)} f_{u_q(\alpha\beta)} \cos\varphi_{u_q(\alpha\beta)}\Big] + V^\beta V^{\beta*}\Big[2\sum_{u_q(\beta\beta)} f_{u_q(\beta\beta)} \cos\varphi_{u_q(\beta\beta)}\Big]\Big\} ;$$

è facile infatti dimostrare che

$$\sum_{u_q(\alpha\beta)} f_{u_q(\alpha\beta)} \cos\varphi_{u_q(\alpha\beta)} = \sum_{u_q(\beta\alpha)} f_{u_q(\beta\alpha)} \cos\varphi_{u_q(\beta\alpha)} .$$

Per il calcolo di $\bar{I}_q$ $(q \geqslant s-1)$ osserveremo anzitutto che la correlazione tra le funzioni di frangia corrispondenti alle possibili complessioni di traslazioni successive è esprimibile mediante l'introduzione di una matrice Q, analoga alla Q' definita dalla (A-26); due vettori opportunamente definiti ci consentiranno poi di esprimere in modo appropriato i prodotti dei fattori di struttura per le funzioni di frangia corrispondenti.

Sia V_1 il vettore riga così costituito:

$$(18) \qquad V_1 = \Big|\,\overbrace{V^\alpha V^\alpha \dots \dots \dots \dots V^\alpha}^{(n_{(s-1)}(\alpha\alpha) + n_{(s-1)}'\alpha\beta))\ \text{termini}}\ \ \overbrace{V^\beta V^\beta \dots \dots \dots \dots V^\beta}^{(n_{(s-1)}'\beta\alpha) + n_{(s-1)}'(\beta\beta))\ \text{termini}}\,\Big| ;$$

V_2 il vettore riga così costituito:

$$(19) \quad V_2 = \left| \overset{(n_{(s-1)}(\alpha\alpha))\ \text{termini}}{\overline{V^\alpha\ .\ .\ .\ .\ .\ .\ V^\alpha}} \quad \overset{(n_{(s-1)}(\alpha\beta))\ \text{termini}}{\overline{V^\beta\ .\ .\ .\ .\ .\ .\ V^\beta}} \right. \cdot$$

$$\left. \cdot\ \overset{(n_{(s-1)}(\beta\alpha))\ \text{termini}}{\overline{V^\alpha\ .\ .\ .\ .\ .\ .\ V^\alpha}} \quad \overset{(n_{(s-1)}(\beta\beta))\ \text{termini}}{\overline{V^\beta\ .\ .\ .\ .\ .\ .\ V^\beta}} \right|;$$

Siano quindi F e Q le matrici quadrate, dello stesso ordine (n^{s-1}) dei vettori già definiti:

$$(20) \qquad F = |F_{uu'}| = |f_u\, \delta^{(uu')}|, \qquad Q = |Q_{uu'}|.$$

Ciò premesso, si avrà:

$$(21) \qquad I_q(q \geqslant s-1) = 2\ \mathrm{Re}\,\{(N_3-q)V_1 F Q^{q-s+1} \widetilde{V}_2^*\}.$$

Il valore dell'intensità media diffratta per strato diventerà pertanto (vedi formula (17))

$$(22) \qquad \bar{I}_{\mathrm{Av}} = \frac{\bar{I}}{N_3} = \frac{1}{N_3}\sum_{q=0}^{s-2}\bar{I}_q + \frac{1}{N_3}\sum_{q=s-1}^{N_{3}-1}\bar{I}_q = \frac{1}{N_3}\sum_{q=0}^{s-2}\bar{I}_q + 2\ \mathrm{Re}\,\left(V_1 F(E-Q)^{-1}\widetilde{V}_2^*\right).$$

Questa formula sarà valida, anche nel caso presente, solo in condizioni di convergenza espresse dalla (18) del capitolo precedente.

SUMMARY

In the present paper, which constitutes the theoretical part of a series of three notes on the same topic, a new mathematical treatment of the calculation of the X-ray diffracted intensity by monodimensionally disordered structures is given. This method, which generally allows very simple calculations, utilizes the matrix method in a form that differs very much from that already discussed by HENDRICKS and TELLER [1]: first of all, because here we consider, from a probabilistic point of view, the succession of the possible translation vectors between adjacent layers, instead of the succession of the layers in different absolute positions; in the second place, because here any matrix diagonalization is avoided; lastly because the present method is extended to the most general case of any value of s (order of influence in the layers succession) and of different layer types.

Canonical Variables in Relativistic Quantum Theory (*).

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Summary. — A simple general method is presented for the explicit determination of the operator corresponding to « position » in any relativistic one-particle theory. It works in all the well known cases.

1. – Introduction.

Several papers have appeared during the last decade on the subject of « localized states » in the framework of relativistic one-particle theory. Notable among them are the works of Newton and Wigner [1], and Foldy and Wouthuysen [2]. Recently it has also been demonstrated that one could talk about « localized states » even for particles of zero mass and spin one or greater in the form of a « front » description, thereby providing a unified description of « position » in relativistic quantum theory [3].

The object of this paper is to present a method for the determination of « position » in the sense of a generalization of Newton and Wigner [1], which is capable of yielding results independent of any relativistic wave equation.

2. – Nature of the method: Dirac equation.

We shall illustrate the method by considering the Dirac equation for a spin $\frac{1}{2}$ particle with the *non-singular* Hamiltonian

$$(1) \qquad H = (\alpha p) + \beta m .$$

---

[1] T. D. Newton and E. P. Wigner: *Rev. Mod. Phys.*, **21**, 400 (1949).
[2] L. Foldy and S. A. Wouthuysen: *Phys. Rev.*, **78**, 29 (1950).
[3] R. Acharya and E. C. G. Sudarshan: *Journ. Math. Phys.*, **1**, 532 (1960).

The method is easily adaptable to higher spin equations, as we shall see shortly.

We shall call an object q_i a « position operator » if it obeys the following three postulates:

(a) $$[q_i, q_j] = 0 \,,$$

(b) $$[q_i, p_j] = i\delta_{ij} \,,$$

(c) $$[[q_i, H], H] = 0 \,.$$

The last postulate implies that the acceleration of a « free particle » vanishes identically. (Recall that an « elementary system » is one which provides an irreducible representation of the 3-dimensional inhomogeneous Euclidean group) [3].

If we now notice that the position operator q_i is a « polar vector » and is arbitrary up to a unitarity transformation in p^2 [4], we can write

(2) $$q_i = x_i + f_i(\boldsymbol{\alpha}, \beta, p^2) \,.$$

---

[4] Within the framework of classical Dirac theory, it appears that all position operators with commuting components are trivially related to the Foldy-Wouthuysen-Newton-Wigner-Pryce « mean » position operator. The argument is as follows and was kindly pointed out to us by Professor SUDARSHAN.

Let us work in the C-representation of the Dirac theory and let β be a position operator satisfying all the three postulates mentioned in the very beginning and further it should satisfy:

$$\beta \xi_j - \xi_j \beta \,.$$

i.e., the « evenness » condition. It is also necessary to keep in mind (although not explicitly mentioned in the text) that the position operator should be *hermitian, i.e.* $\xi_j^+ = \xi_j$. Now, the only « even » operators in the C-representation are x, σ, p and β. The most general choice of ξ is (remember ξ is a polar object)

$$\boldsymbol{\xi} = \boldsymbol{x} + \left\{ a_1(p^2) + \beta a_2(p^2) \right\} \boldsymbol{p} + \left\{ b_1(p^2 + \beta b_2(p^2) \right\} (\boldsymbol{\sigma} \times \boldsymbol{p}) \,.$$

(Notice that $[\boldsymbol{\xi} - \boldsymbol{x}, \boldsymbol{p}] = 0$ so that $\boldsymbol{\xi} - \boldsymbol{x}$ should *not* contain any dependence on x.) Terms like $c(p^2)\sigma$ are not allowed because they are axial vectors. $a(p^2)$ etc., are invariant functions of $p \cdot p = p^2$. Now compute the commutator $[\xi_i, \xi_j]$. By elementary methods, one finds that $[\xi_i, \xi_j] = 0$ only if $b_1(p^2) = 0 = b_2(p^2)$.

Thus the most general form for a commuting ξ is

$$\boldsymbol{\xi} = \boldsymbol{x} + \left\{ a_1(p^2) + \beta a_2(p^2) \right\} \boldsymbol{p} \,.$$

The idea of the method is to start from the three postulates and « work backwards. »

$$(2a) \qquad\qquad [q_i, H] = -iV_i ,$$

where V_i is the « velocity operator », *i.e.*,

$$(3) \qquad\qquad q_i - H^{-1}q_iH = iH^{-1}V_i .$$

Trivial manipulation of (3) yields (remember postulate (c))

$$(4) \qquad\qquad H^{-1}q_iH - H^{-2}q_iH^2 = iH^{-1}V_i .$$

From (3) and (4)

$$(5a) \qquad\qquad q_i - H^{-2}q_iH^2 = 2iH^{-1}V_i ,$$

$$(5b) \qquad\qquad H^2 = p^2 + m^2 = E^2 ,$$

$$(5c) \qquad\qquad q_i - (p^2 + m^2)^{-1}q_i(p^2 + m^2) = 2iH^{-1}V_i ;$$

observe that

$$(6) \qquad\qquad [q_i, H] = -iV_i = -i\frac{p_iH}{E^2} .$$

From (1), (2), (5c) and (6):

$$(7) \qquad\qquad [\hat{H}, f_i] = i\left\{ \frac{p_i \hat{i}}{E^2} + \frac{\alpha_i}{E} \right\} .$$

One can see that ξ and x are related by means of the unitary transformation

$$U = \exp\left\{ \frac{1}{2i} \int d(p^2)[a_1(p^2) + \beta a_2(p^2)] \right\} .$$

i.e.,

$$\xi = UxU^{-1} .$$

This is of course consistent with all the properties demanded of ξ since all these are invariant under unitary transformations which are invariant under space rotations. Notice that this unitary equivalence was of course to be expected since ξ was defined only in terms of algebraic properties like commutation relations. This certainly brings in a certain amount of arbitrariness in the configuration space description of localized states but in no way alters the momentum representation of NEWTON and WIGNER ([1]).

where

$$\hat{H} \quad \frac{H}{E}; \qquad \hat{H}^2 = 1 .$$

The solution unique up to a unitary transformation in p^2 of (7) is immediate:

$$(8) \qquad q_i = x_i + \frac{i}{2} H^{-1}\alpha_i .$$

Note that the position operator so obtained satisfies postulates (b) and (c) but not (a): its components do *not* commute. This is easily taken care of. The «trick» is to find a special solution K_i which when added to (8) makes things «all right». This is done in the Appendix. The final outcome is that the position operator q_i obtained by this simple method has the following form:

$$(9) \qquad q_i = x_i + \frac{i\beta\alpha_i}{2E} - \frac{i\beta(\alpha p)p_i + (\sigma \times p)_i p}{2E(E + m)p} ,$$

which is identical with Newton-Wigner-Foldy-Wouthuysen's «position».

3. – Duffin-Kemmer equation: spin 1 and finite mass.

The extension to higher spin, finite mass case is straightforward. The «trick» here is to first project out the singular part of the Hamiltonian before working things out.

For purposes of comparison with the previous literature ([5]), let us consider the «C-representation» Hamiltonian for the Kemmer equation ([6]):

$$(10) \qquad H = \beta_4 E .$$

The operator which projects out the zero eigenvalue of β_4 is

$$(11) \qquad \Lambda_0 = 1 - \beta_4^2 + \frac{1}{m} \beta_4(\alpha_K p_K) ,$$

([5]) M. H. L. PRYCE: *Proc. Roy. Soc.*, A **195**, 62 (1948); L. M. GARRIDO and P. PASCUAL: *Nuovo Cimento*, **12**, 181 (1959). See also M. A. MELVIN: *Rev. Mod. Phys.*, **32**, 477 (1960).
([6]) N. KEMMER: *Proc. Roy. Soc.*, A **173**, 91 (1939).

where

$$(12) \quad \begin{cases} \alpha_K = -iS_{K_4}\,, \\ S_{Kl} = i(\beta_K\beta_i - \beta_i\beta_K)\,. \end{cases}$$

The Hamiltonian in the space orthogonal to the null space β_4 is

$$(13) \qquad H' = H(1 - \Lambda_0) = \beta_4 E\left(1 - \frac{1}{m}\,\beta_K p_K\right),$$

where use has been made of $\beta_i\beta_K$, $\beta_i = 0$, $i \neq K$.

This gives

$$(14) \quad q_i = x_i - \frac{i}{2}\,H'^{-1}\left(\frac{\partial H'}{\partial p_i}\right) + K_i = x_i + \frac{\beta_i}{2m} + \frac{1}{2E^2 m}\,(\beta p)p_i - \frac{i}{2E^2}\,p_1 + K_i\,.$$

One could proceed to calculate K_i in the same fashion as in the case of the Dirac equation (see Appendix) but it is easier to see by inspection that K_i is the « spin » contribution, namely $(1/Em)(P \times S)_i$. The final result in the case of spin one and finite mass:

$$(15) \qquad q_i = x_i + \frac{\beta_i}{2m} + \frac{1}{2E^2 m}\,(\beta p)p_i - \frac{i}{2E^2}\,p_i + \frac{1}{Em}\,(P \times S)_i\,,$$

coincides with that given by PRYCE, apart from trivial factors. Notice the occurrence of the non-hermitian term in p_i, which also occurs in the expression given by PRYCE. It can be ignored if one chooses proper normalization for the wave function, as pointed out by PRYCE himself ([5]).

We have here chosen to work with the Kemmer equation ([6]). One could equally well have worked with the « reduced » form of this equation, given by SAKATA and TAKETANI ([7]) and CASE ([8]). Here again, our method yields results identical with that of CASE.

4. – The photon equation: spin 1 and zero mass.

We have restricted ourselves in the previous sections to particles of finite mass. In the case of vanishing mass and spin one or greater, it has been demonstrated elsewhere ([1]) that « cartesian position » does not exist. One sees

([7]) M. TAKETANI and S. SAKATA: Proc. Phys. Math. Soc. (Japan), 22, 757 (1940).
([8]) K. M. CASE: Phys. Rev., 95, 1323 (1954).

by introducing generalized co-ordinates which possess well-defined conjugate momenta, that one can still talk of « position ». In fact, in the « front » description, the three indices which completely specify the « position » of an « elementary » particle are the « distance traversed » (ϱ) and the « unit normal » ($\widehat{p}$),

$$
(16) \qquad
\begin{cases}
\varrho = \tfrac{1}{2}(\boldsymbol{x} \cdot \widehat{p} + \widehat{p} \cdot \boldsymbol{x}) \,, \\[2mm]
\widehat{p} = \boldsymbol{p}/|p| \,.
\end{cases}
$$

The conjugate « momenta » are respectively the magnitude of the momentum $|p| = \sqrt{\sum_i p_i^2}$ and the total angular momentum, $\boldsymbol{J}$.

Before we proceed to obtain the above result by the methods of this paper, we have to cast Maxwell's equations for the electromagnetic field into a Schrödinger equation for the photon. Following SUDARSHAN and GOOD [9,10]), one can show that Maxwell's equations,

$$
(17) \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \qquad \partial_\mu F_{\mu\nu} = 0 \,,
$$

where A_μ is the four-vector potential of the electromagnetic field and can be

----

[9] E. C. G. SUDARSHAN (unpublished). See also R. H. GOOD Jr.: *Phys. Rev.*, **105**, 1914 (1957).

[10] There exists another method of reduction. One starts off from the Kemmer equation for a particle of spin 1, in the Hamiltonian form:

$$
H\psi \equiv (\alpha_K p_K + \beta_4 m)\psi = i \frac{\partial \psi}{\partial t} \,.
$$

By means of the pseudounitary transformation [11]

$$
\psi' = U\psi; \qquad U = \exp\left[i \frac{\beta_K p_K}{p} \operatorname{tg}^{-1}\left(\frac{m}{p}\right) \right],
$$

one obtains the following Hamiltonian:

$$
H^E = \frac{\alpha_K p_K}{p} E \,.
$$

Let

$$
\psi_0 = \tfrac{1}{2}(1 + \beta_5)\psi \,.
$$

Then, ψ_0 satisfies, in the zero mass limit:

$$
H\psi_0 = \boldsymbol{s} \cdot \boldsymbol{p} \; \psi_0 \,.
$$

[11] Compare S. K. BOSE, A. GAMBA and E. C. G. SUDARSHAN: *Phys. Rev.*, **113**, 1661 (1959).

cast into the form

$$i \frac{\partial \psi}{\partial t} = \mathbf{s} \cdot \mathbf{p} \psi \,,$$

with the supplementary condition

(18) $\partial_K \psi_K = 0 \,,$

where

$$\psi_j \sim [-4\nabla^2]^{-\frac{1}{2}} \{E_j + i\varepsilon_{ikl}\partial_k A_l\} \,,$$

$$(s_k)_{jl} = i\varepsilon_{jkl} \,.$$

$$[s_i, s_j] = i\varepsilon_{ijk}s_k \,.$$

The supplementary condition ensures that the photon is always transversely polarized with just two independent polarizations.

We now write in anticipation

(19) $\xi_\gamma = \frac{1}{2}(\mathbf{x} \cdot \widehat{p} + \widehat{p} \cdot \mathbf{x}) + f(|p|, \widehat{p} \cdot \mathbf{s}, s^2) \,; \quad \widehat{\xi} = \widehat{p} \,,$

where ξ_γ is the radial component of the position operator and $\widehat{\xi}$ its direction. The equation

(20) $[\xi_\gamma, H] = -i \, V = \frac{H}{P} \,,$

still holds.

It is possible to get rid of the second term in the equation for ξ_γ in (19) by means of a unitary transformation. This transformation is explicitly given by

(21) $U = \exp \left[i \int f(|p|, \widehat{p} \cdot \mathbf{s}, s^2) \, \mathrm{d}\,|p| \right] \,,$

where the integration indicated is carried out regarding the variables $p, \widehat{p} \cdot \mathbf{s}, s^2$ as independent variables and preserving the order of non-commuting factors. (21) can easily be verified using the commutation relations

$$[\tfrac{1}{2}(\mathbf{x} \cdot \widehat{p} + \widehat{p} \cdot \mathbf{x}), |p|] = i \,; \quad [\widehat{p}, \tfrac{1}{2}(\mathbf{x} \cdot \widehat{p} + \widehat{p} \cdot \mathbf{x})] = 0 \,.$$

We still have to show that these relations continue to hold when the Hilbert space in question is restricted by the supplementary condition (18). For this

purpose, introduce the projection operator Λ given by

$$(22) \qquad \Lambda\boldsymbol{\psi} = \widehat{p} \times (\widehat{p} \times \boldsymbol{\psi}) \,,$$

where $\boldsymbol{\psi}$ is the « vector » wave function. Λ clearly projects out that part of the wave function space which satisfies (18). In this « projected » space, our position operator reduces to the form

$$(23) \qquad \boldsymbol{\xi}' = \Lambda\boldsymbol{\xi}\Lambda \,.$$

But since Λ commutes with both the angular and the radial part of $\boldsymbol{\xi}$, $([\xi_\gamma, \widehat{p}] = 0)$, ξ' reduces to $\xi\Lambda^2$. Recalling that $\Lambda^2 \equiv 1$, in the « projected » space, our assertion follows.

5. – Conclusion.

We have presented a simple method for computing « position operators » in the framework of relativistic one-particle theory. The method works for particles of all masses and all spins.

<p style="text-align:center">* * *</p>

<p style="text-align:center">APPENDIX</p>

Dirac equation for spin $\frac{1}{2}$.

One can easily see that the most general operator K which depends on $\boldsymbol{p}, \hat{\boldsymbol{a}}, \beta$ and m, and which commutes with H is of the form

$$(A\text{-}1) \qquad \boldsymbol{K} = \boldsymbol{T} - \frac{\beta}{2m} \left[(\alpha p), \boldsymbol{T} \right] \,,$$

where T is also a function of p, α, β and m, and further commutes with β. It is easy to see that K commutes with H by explicitly taking the commutator. The fact that this is the most general form for K can be seen by expansion in powers of $1/m$ and requiring that it should commute with H.

The most general form of T is seen to be

$$(A\text{-}2) \qquad \boldsymbol{T} = \beta\boldsymbol{p}Z_0(p^2) + (\boldsymbol{\sigma} \times \boldsymbol{p})Z_1(p^2) + \beta(\boldsymbol{\sigma} \times \boldsymbol{p})Z_2(p^2) \,,$$

substituting this into (A-1), and requiring that $[q_i, q_j] = 0$ one can easily determine Z_0, Z_1, Z_2.

The final result is

$$q_i = x_i + \frac{i\beta\alpha_i}{2E} - \frac{i\beta(\alpha p)p_i + (\boldsymbol{\sigma} \times \boldsymbol{p})_i p}{2E(E+m)p} ,$$

which is the Newton-Wigner-Foldy-Wouthuysen « position ».

RIASSUNTO (*)

Si presenta un metodo generale semplice per la determinazione esplicita dell'operatore corrispondente alla « posizione » in qualsiasi teoria relativistica a una particella. Il metodo si adatta a tutti i casi ben conosciuti.

The Definition of Electromagnetic Radiation (*).

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Summary. — Momentum and energy of electromagnetic radiation emitted from a charge are Lorentz-covariantly defined. The non-local measurement of a Lorentz invariant, the rate of radiation energy emission is proposed as a necessary and sufficient criterion for the emission of radiation at a given time. This measurement is a field measurement which can be performed at *any* distance from the charge and need not be made in the wave zone. It can be regarded as an operational definition of radiation in terms of rate of energy emission. These results permit a simple derivation of the equation of motion of a charged particle which avoids the self-energy difficulties. A question concerning the principle of equivalence and radiation in uniformly accelerated motion is answered.

1. – Statement of the problem.

Despite the general consensus of opinion that classical electromagnetic theory is well understood, there seem to be a number of problems which are repeatedly encountered, and which raise fundamental questions. One such problem might be broadly labelled « the definition of radiation ». It is characterized by the following questions:

1) is radiation à Lorentz-invariant phenomenon? If one inertial observer sees radiation, do all inertial observers see it, such that there is no Lorentz transformation by which radiation can be « transformed away »?

2) In order to tell whether a system radiates, is it necessary to go to very large distance (compared to the wavelength)? And if not, what measurements at small distance can determine the presence of radiation? Is this measurement unambiguous? Is it Lorentz-invariant?

F. ROHRLICH

The first question was raised recently by SYNGE [1]. It was answered to a large extent by SCHILD [2]. His invariance proof is basic to our later arguments.

The second question has recently played a crucial role in certain arguments concerned with the principle of equivalence [3]. In these arguments it was taken for granted that the presence or absence of radiation cannot be determined at arbitrarily small distance from an accelerated point charge. Since it will be shown in Section 2 that this is *not* the case, some remarks will have to be made concerning the principle of equivalence.

Finally, the covariant formulation of radiation permits a simple derivation of the equation of motion.

2. – The momentum four-vector of radiation.

Given a point charge e of known trajectory $z^\mu(\tau)$, where τ is the proper time, with velocity $v^\mu = dz^\mu/d\tau$ and acceleration $a^\mu = dv^\mu/d\tau$. The electromagnetic fields produced by the charge's motion, $F^{\mu\nu}$, are measured at a point P with co-ordinates x^μ. Assuming for definiteness retarded fields (advanced fields would do just as well), the point P is uniquely correlated with a point Q on the world-line $z^\mu(\tau)$, such that $R^\mu \equiv x^\mu - z^\mu$ is a null-vector, $R_\mu R^\mu = 0$. The choice of retarded fields specifies furthermore $R^0 > 0$ (we use a Minkowski metric $\eta^{\mu\nu}$ with signature $+2$, and we measure time in light-seconds).

The invariant distance

$$(2.1) \qquad\qquad \varrho \equiv -v_\lambda R^\lambda > 0$$

can now be defined. It is equal to the spatial distance between P and Q in the special inertial system S_q in which Q is instantaneously at rest. v^μ will always refer to the retarded point Q.

We can define a space-like vector n^μ

$$(2.2) \qquad\qquad n^\mu \equiv \frac{1}{\varrho} R^\mu - v^\mu ,$$

which has the properties

$$(2.3) \qquad\qquad n_\lambda n^\lambda = 1 , \qquad n_\lambda v^\lambda = 0 ,$$

[1] J. L. SYNGE: *Relativity: The Special Theory* (Amsterdam and New York, 1956).
[2] A. SCHILD: *Journ. Math. Analysis and Appl.*, **9**, 127 (1960).
[3] T. FULTON and F. ROHRLICH: *Ann. Phys.*, **9**, 499 (1960).

as can easily be verified. R^μ can therefore be written

$$(2.4) \qquad\qquad R^\mu = \varrho(n^\mu + v^\mu) \; .$$

In this notation the Liénard-Wiechert potentials are [4] (Gaussian units),

$$(2.5) \qquad\qquad A^\mu(x) = ev^\mu/\varrho \; .$$

So far only the null-vector character of R^μ was specified. If retarded or advanced potentials are to be used, one writes

$$(2.6) \qquad\qquad R^\mu = (\pm R, \boldsymbol{R}) \; , \qquad R = |\boldsymbol{R}| \; ,$$

where the upper (lower) sign refers to the retarded (advanced) case. As mentioned above, we shall use retarded potentials in the following.

From (2.5) one obtains for the retarded fields the antisymmetric tensor [4]

$$(2.7) \qquad F^{\mu\nu}(x) = -\frac{2e}{\varrho^2}\, n^{[\mu}v^{\nu]} - \frac{2e}{\varrho}\, [v^{[\mu}a^{\nu]} + n^{[\mu}(v^{\nu]}a_n + a^{\nu]})] \; ,$$

where $a_n = a_\lambda n^\lambda$ and the antisymmetrization symbol is as usual defined as

$$a^{[\mu}b^{\nu]} = \tfrac{1}{2}(a^\mu b^\nu - a^\nu b^\mu) \; .$$

The first term in (2.7) is the (generalized) Coulomb field, the second term, which vanishes if and only if the acceleration vector vanishes, is the radiation field, by definition.

The electromagnetic energy tensor is defined by

$$(2.8) \qquad\qquad 4\pi T^{\mu\nu} = F^{\mu\lambda}F_\lambda{}^\nu + \tfrac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} \; .$$

Inserting (2.7) into this expression one finds the energy tensor at a point P due to the field emission at the retarded point Q on the world line $z^\mu(\tau)$ of a point charge e,

$$(2.9) \quad 4\pi T^{\mu\nu}(x) = \frac{e^2}{\varrho^4}(n^\mu n^\nu - v^\mu v^\nu - \tfrac{1}{2}\eta^{\mu\nu}) + \frac{2e^2}{\varrho^3}[a_n u^\mu u^\nu - u^{(\mu}(v^{\nu)}a_n + a^{\nu)})] +$$

$$+ \frac{e^2}{\varrho^2} u^\mu u^\nu(a_n^2 - a_\lambda a^\lambda) \; .$$

The velocity and acceleration vectors always refer to the point Q, and

(2.10) $u^\mu \equiv n^\mu + v^\mu \,.$

Consider now a line element $d\tau$ of the world line $z''(\tau)$. This is the infinitesimal invariant distance of two points Q and Q' on this line. If one draws the forward light cones from Q and Q' and an arbitrary time-like cylinder surrounding the world line, the light cones will cut a surface Σ from this cylinder (see Fig. 1). Σ is a three dimensional surface with space-like surface normal. It forms a band around the world line, which lies between the two light cones.
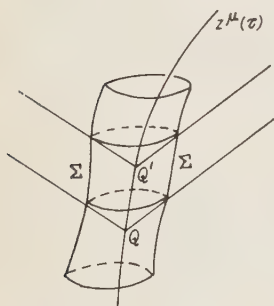
One defines the four quantities

(2.11) $dP^\mu = - \int\limits_{\Sigma} T^{\mu\nu} d^3\sigma_\nu \,.$

Fig. 1.

In a recent paper ([2]) SCHILD has shown that the conservation law

$$\partial_\mu T^{\mu\nu} = 0$$

which holds in charge-free regions implies that

(2.12) $d\overline{P}^\mu \equiv \lim_{\Sigma \to \infty} dP^\mu$

is independent of the surface and is consequently a four-vector. The notation $\Sigma \to \infty$ is symbolic to indicate that the point of closest approach of Σ to the world line, $\varrho_{\min}$, is taken to the limit. In going to the limit, Σ is moved outwards always remaining between the two light cones; the limit therefore corresponds to a *simultaneous* limit in space-like and time-like directions.

Schild's proof is simply an application of Gauss' theorem to the above conservation law. But it is essential that the four-volume involved is a doubly connected (annular) region between two surfaces of the type Σ, so that one *never* crosses or even reaches the world line $z''(\tau)$ on which the fields are singular.

Let us take for Σ the special surface Σ_0 which is a sphere in the inertial system in which Q is instantaneously at rest,

(2.13) $d^3\sigma^\mu = n^\mu d\tau \varrho^2 d\Omega \,.$

From (2.12) and (2.9) we find

$$(2.14) \qquad \frac{\mathrm{d}\overline{P}^\mu}{\mathrm{d}\tau} = \frac{e^2}{4\pi} \lim_{\varrho \to \infty} \int_{\Sigma_0} u^\mu (a_\lambda a^\lambda - a_n^2) \, \mathrm{d}\Omega = \tfrac{2}{3} c^2 a^\lambda a_\lambda v^\mu \, .$$

This four-vector is the total rate of energy and momentum emission in form of radiation. We note that it is a *time-like* vector. The invariant rate of energy emission is

$$(2.15) \qquad \mathscr{R} \equiv - v_\mu \, \frac{\mathrm{d}\overline{P}^\mu}{\mathrm{d}\tau} = \frac{2}{3} e^2 a_\lambda a^\lambda \, .$$

This establishes the Lorentz invariance of electromagnetic radiation. $\mathscr{R}$ is non-negative and vanishes if and only if the four-vector of acceleration vanishes.

3. – The criterion for radiation.

With the special surface Σ_0 one can compute the rate of energy and momentum which crosses this surface in the direction n^μ per unit solid angle,

$$(3.1) \qquad \frac{\mathrm{d}P^\mu}{\mathrm{d}\tau \, \mathrm{d}\Omega} = - \frac{e^2}{8\pi} \frac{n^\mu}{\varrho^2} + \frac{e^2}{4\pi\varrho} \, (a^\mu - a_n n^\mu) + \frac{e^2}{4\pi} \, u^\mu (a_\lambda a^\lambda - a_n^2) \, .$$

This quantity is obviously a four-vector. The first two terms are both space-like vectors and may be interpreted as the Coulomb four-momentum and the cross term between Coulomb and radiation fields. The last term (cf. eq. (2.10)) is a *null* vector and describes pure radiation. *It is independent of ϱ, but does depend on the direction n^μ*. The null character of the last term corresponds to the momentum four-vector of a particle of zero mass (photon).

Eq. (3.1) depends only on the component of the acceleration which is orthogonal to n^μ, since

$$(a^\lambda - a_n n^\lambda)(a_\lambda - a_n n_\lambda) = a^\lambda a_\lambda - a_n^2 \, .$$

In the instantaneous rest system of Q, S_ϱ, the momentum rate per unit solid angle is

$$(3.2) \qquad \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t \, \mathrm{d}\Omega} = - \frac{e^2}{8\pi} \frac{\hat{n}}{R^2} + \frac{e^2}{4\pi} \frac{\boldsymbol{a}_\perp}{R} + \frac{e^2}{4\pi} \, \hat{n} a_\perp^2 \, ,$$

where the separation

$$a = a_\parallel + a_\perp \,, \qquad a_\parallel = a \cdot \hat{n}\hat{n} \,, \qquad a_\perp = |a \times \hat{n}|$$

is made relative to $\hat{n}$. At small distances the Coulomb field dominates. It gives rise to a spherically symmetric rate of momentum change directed towards the source. At intermediate distances the cross term contributes a vector in the direction $a_\perp$ which lies in a plane orthogonal to n. At large distances the radiation field dominates and yields the Poynting vector in the direction $\hat{n}$, *i.e.* away from the source.

The energy rate per unit solid angle in S_ϱ is

(3.3)
$$\frac{\mathrm{d}W}{\mathrm{d}t\,\mathrm{d}\Omega} = \frac{e^2}{4\pi}\,a_\perp^2 \,, \qquad\qquad (\text{in } S_\varrho)\,.$$

Thus, only the pure radiation term contributes to the *energy* rate.

One now observes that the two space-like terms in (3.1) are both orthogonal to the velocity vector v^μ. Thus,

(3.4)
$$-v^\mu\frac{\mathrm{d}P^\mu}{\mathrm{d}\tau\mathrm{d}\Omega} = \frac{e^2}{4\pi}\,(a_\lambda a^\lambda - a_n^2)\,,$$

which is the invariant generalization of the energy rate (3.3). It is the invariant formulation of Poynting's formula for the radiation energy rate per unit solid angle.

While (3.1) is a four-vector, its integral over the angles is no longer a four-vector unless the limit $\Sigma \to \infty$ is taken, in accordance with (2.12). However, (3.4) is independent of ϱ, so that

$$\mathscr{R} = -v_\mu\frac{\mathrm{d}\overline{P}^\mu}{\mathrm{d}\tau} = -v_\mu\frac{\mathrm{d}P^\mu}{\mathrm{d}\tau}\,,$$

or, with (2.11) and (2.13),

(3.5)
$$\mathscr{R}(\tau) = -\int v_\mu T^{\mu\nu}n_\nu\varrho^2\,\mathrm{d}\Omega\,.$$

It follows that the invariant radiation energy rate can be expressed as an integral over the surface Σ_0 (sphere in the system S_ϱ) which is *independent* of ϱ. The integral in (3.5) gives the invariant result without need to go to the limit $\Sigma_0 \to \infty$.

This result permits one to establish a criterion for testing whether a charge is emitting radiation at a given instant, by measuring the fields only, and without

having to do so at a distance large compared to the emitted wave length. This criterion for radiation can be expressed as follows.

Criterion for radiation: Given the world line of a charge and an arbitrary instant τ_0 on it. Consider a sphere of *arbitrary radius* r in the inertial system S_ϱ which is the instantaneous rest system of the charge at the proper time τ_0 (time t_0 in S_ϱ) and at whose center the charge will be at that instant; measure the electromagnetic fields $F^{\mu\nu}$ on Σ_0 at the time $t_0 + r$; evaluate the integral

$$\mathscr{R}(\tau_0) = \int T^{0k} n_k r^2 \, \mathrm{d}\Omega = \int_{\Sigma_0} \boldsymbol{S} \cdot \hat{n} \, \mathrm{d}^2\sigma \,, \qquad \text{in } S_\varrho \,,$$

where $\boldsymbol{S}$ is the Poynting vector. The value of this integral is the invariant rate of radiation energy emission $\mathscr{R}$ at time t_0 and vanishes if and only if the charge did not radiate at that instant.

Obviously, this criterion is also an operational definition of the invariant radiation energy rate.

Several remarks are in order at this point. First, the measurement is carried out in an inertial system (S_ϱ) and the invariance of $\mathscr{R}$ refers to *Lorentz* transformation only. An accelerated observer may very well measure a *different* rate (or no rate at all). Secondly, the possibility of ascertaining whether a charge radiates without having to go to the wave zone is actually a logical necessity, since the radiation emitted will in general contain Fourier components of all wave lengths, so that one would never be in the wave zone of *all* waves emitted. Thirdly, the measurement is to be made over the surface of a sphere (Σ_0). Thus, although the radius of the sphere can be chosen arbitrarily, it can never be zero. The measurement is therefore intrinsically *non-local*.

4. – The equation of motion.

The above derivation was entirely based on the Maxwell-Lorentz equations. But it permits one to go far beyond these electromagnetic field equations. One can, in fact, « almost » derive the equations of motion of a charged particle from the above results, making use only of the field equations and the conservation laws.

Consider a particle of mass m and charge e. An external force F^μ is acting on it. Like all forces in special relativity, this is a space-like vector orthogonal to the velocity of the particle,

$$(4.1) \qquad\qquad F^\mu v_\mu = 0 \,.$$

If the law of conservation of momentum and energy is to hold during every interval of proper time $\mathrm{d}\tau$, the external force must account for the (reversible)

increase in some momentum vector p^μ as well as for the four-vector of energy and momentum lost irreversibly in the form of radiation,

$$(4.2) \qquad\qquad F^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} + \mathscr{R}v^\mu \,,$$

according to (2.14) and (2.15). Note that p^μ *cannot* be identified with the kinetic momentum mv^μ, because multiplication of (4.2) with v^μ would yield $\mathscr{R} = 0$. This has to do with the fact that the radiation-loss vector is time-like.

The « ansatz » (4.2) so far contains only one well-defined term, *viz.* the last one. The other two terms are written down first of all because one « guesses » a certain structure for an equation of motion. This structure is the covariant generalization of the statement of energy conservation (corresponding to $\mu = 0$) per unit time. Since (4.1) implies that F^μ is of the form

$$F^\mu = (\gamma \boldsymbol{v} \cdot \boldsymbol{F}, \gamma \boldsymbol{F}) \,,$$

the statement is that the work per unit proper time done by the impressed force consists of two parts, one which is a total time derivative, and the other one which is *not* a total time derivative. This is a separation into reversible and irreversible energy change. The reversible energy change and its covariant generalization will be called « inertial terms ». These terms are *uniquely* defined in (4.2) as the difference between F^μ and the radiation term. It is assumed that this difference is a total derivative, *i.e.* that *radiation is the only dissipative process*.

This establishes the formal structure of (4.2). But a physical specification of F^μ and the inertial term is still lacking. We now specify that F^μ should not contain terms which depend *only* on the motion (v^μ, a^μ, etc.) and the characteristic properties of the particle (m, e, etc.). Such terms would have to be « on the right-hand side » of this equation. Thus, a term of the form λa^μ cannot be tolerated in F^μ, but (if λ is a constant) it belongs into $\mathrm{d}p^\mu/\mathrm{d}\tau$. These considerations imply that one must regard the equation of motion (4.2) as « already renormalized » in the language of field theory.

In fact, the force F^μ is assumed to be entirely due to the external fields. (See (4.5) and (4.9) below.)

Combining (4.1) and (4.2) one finds for the rate of four-momentum changes

$$v_\mu \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \mathscr{R} = \tfrac{2}{3} e^2 a_\lambda a^\lambda = -\tfrac{2}{3} e^2 v_\lambda \frac{\mathrm{d}a^\lambda}{\mathrm{d}\tau} \,,$$

or

$$(4.3) \qquad\qquad \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = -\tfrac{2}{3} e^2 \frac{\mathrm{d}a^\mu}{\mathrm{d}\tau} + v_\perp^\mu \,,$$

where $v_\perp^\mu$ is a vector orthogonal to v^μ, which must be expressible in terms of v^μ and its derivatives and must itself be a total derivative with respect to τ.

The equation of motion (4.2) can now be written

$$(4.4) \qquad r_\perp^\mu = F^\mu + \tfrac{2}{3} e^2 \left(\frac{\mathrm{d} a^\mu}{\mathrm{d}\tau} - a_\lambda a^\lambda v^\mu \right).$$

Since F^μ is in general not a purely electromagnetic force,

$$(4.5) \qquad F^\mu = F_{\mathrm{elm}}^\mu + F_{\mathrm{neutral}}^\mu,$$

this equation must reduce to the equation of motion of a neutral particle,

$$(4.6) \qquad m a^\mu = F_{\mathrm{neutral}}^\mu,$$

in the limit $e \to 0$. (We exclude here for simplicity higher electromagnetic multipole moments.) This requirement, that (4.4) should become (4.6) in the limit $e \to 0$ is very basic and was elsewhere referred to ([5]) as « the principle of undetectability of sufficiently small charge ».

From this requirement one deduces.

$$(4.7) \qquad v_\perp^\mu = m a^\mu + v_\perp'^\mu.$$

The vector $v_\perp'^\mu$ cannot be determined, unless one requires that (4.4) should be a differential equation of order not higher than third order. In that case $v_\perp'^\mu = 0$ and (4.4) becomes

$$(4.8) \qquad F^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau} (m r^\mu - \tfrac{2}{3} e^2 a^\mu) + \mathscr{R} v^\mu.$$

This is the desired equation of motion. From the discussion of (4.2) it is clear that m represents the observed rest mass of the particle and F^μ is the « renormalized » interaction, *i.e.*, the interaction due to external effects only. It is orthogonal to v^μ, retarded, and in general of the form (4.5).

From these remarks follows that F_{elm}^μ must be given by the retarded *external* fields only, for example,

$$(4.9) \qquad F_{\mathrm{elm}}^\mu = e F_{\mathrm{ext}}^{\mu\nu} v_\nu.$$

([5]) F. ROHRLICH: *Ann. Phys.*, **10**, 93 (1961).

The Lorentz force (4.9) can be combined with (4.8) to yield

$$(4.10) \qquad ma^\mu = eF^{\mu\nu}_{\text{ext}}v_\nu + \tfrac{2}{3}\,e^2\left(\frac{\mathrm{d}a^\mu}{\mathrm{d}\tau} - a_\lambda a^\lambda v^\mu\right),$$

which is the Lorentz-Dirac ([6]) equation.

The above derivation of the equations of motion (4.8) from the field equations involved only the conservation law, the principle that the limit $e \to 0$ should result in (4.6), and the simplicity assumption $r^{\mu'} = 0$. The self-energy difficulties have been avoided, but the form of the interaction with the external fields (e.g. (4.9)) is not specified by this derivation.

Finally, it must be remarked that the equations of motion (4.8) or (4.10) are third order equations and are therefore to be supplemented by a suitable asymptotic condition which effectively reduces them to second order equations. At the same time it eliminates the run-away solutions. Together with this condition (4.8) becomes ([5])

$$(4.11) \qquad \frac{\mathrm{d}^2z^\mu(\tau)}{\mathrm{d}\tau^2} = \frac{\exp[\tau/\tau_0]}{\tau_0}\int_\tau^\infty \exp[-\tau'/\tau^0][F^\mu(\tau') - \mathscr{R}(\tau')v^\mu(\tau')]\,\mathrm{d}\tau',$$

where $\tau_0 = \tfrac{2}{3}e^2/m$. Only this equation can be regarded as a true equation of motion, since it is of second order and determines the world-line from the initial position and velocity. Precise conditions for the existence and uniqueness of solutions of (4.11) were given by HALE and STOKES ([7]).

5. – Conclusions.

1) The rate of change (per unit proper time) of field momentum and energy $\mathrm{d}P^\mu/\mathrm{d}\tau\,\mathrm{d}\Omega$ present at a given distance from the charge, and due to fields produced by the charge at a given instant and into a given solid angle, is a four-vector, given by (3.1) and containing ϱ^{-2}, ϱ^{-1}, and ϱ^0 terms. The first two are space-like vectors not transmitting energy like particles of real mass, $m^2 = -p_\lambda p^\lambda$, while the last term is a null vector representing radiation.

2) The total (i.e. integrated over all directions) rate of change of field momentum and energy, $\mathrm{d}P^\mu/\mathrm{d}\tau$, on a given surface Σ surrounding the charge and emitted at a given instant, is *not* a four-vector. The limit $\mathrm{d}\bar{P}^\mu/\mathrm{d}\tau$ obtained

([6]) P. A. M. DIRAC: Proc. Roy. Soc., **167**, 148 (1938).

([7]) J. K. HALE and A. P. STOKES: On some physical solutions of Dirac type equations, preprint, RIAS, Baltimore, Md. (April 1961).

for $\Sigma \to \infty$, however, *is* a four-vector (Schild's theorem). This four-vector is time-like and parallel to the (retarded) velocity four-vector.

3) The invariant total rate of energy emission (an irreversible process) $\mathscr{R}$, defined in (2.15), is a non-negative invariant and vanishes if and only if the acceleration four-vector vanishes. It is due to the radiation fields only. This establishes radiation as a Lorentz-invariant phenomenon and makes the measurement of $\mathscr{R}$ a necessary and sufficient criterion of the presence of radiation.

4) This measurement of $\mathscr{R}$ is a *non-local* one and can be carried out as a field measurement over a suitable sphere of *arbitrary* radius (cf. Section **3**). This establishes that radiation can be detected and measured unambiguously at arbitrary distance from the emitting charge.

5) The relativistic equation of motion of a charge can be derived without encountering self-energy difficulties by assuming

    a) the Maxwell-Lorentz equations;

    b) the Lorentz-invariant conservation laws of energy and momentum;

    c) the principle of undectability of sufficiently small charges;

    d) the equation of motion is at most of third order.

All these assumptions were also used by DIRAC ([6]).

6) The invariance of $\mathscr{R}$ mentioned in (3) refers to Lorentz invariance. Transformation to a non-inertial system does not leave $\mathscr{R}$ invariant. In particular, transformation to a suitably uniformly accelerated system S' will make $\mathscr{R}$ vanish. The proof of this statement follows from the fact that this transformation can be brought about by a conformal transformation ([3]). Such a transformation leaves Maxwell's equations invariant, so that the radiation criterion holds also in the uniformly accelerated, non-inertial, system S'. On the other hand, the acceleration in that system vanishes identically. This establishes that a uniformly accelerated (relative to an inertial system) charge, while radiating at a constant rate $\mathscr{R}$ as seen from an inertial system S, will not be seen to emit radiation by the co-moving (non-inertial) observer S' *at any distance* from the charge ([3]).

Consequently, there is no contradiction between electromagnetic theory and the *principle of equivalence*, even when the latter is referred to constant homogeneous gravitational fields which extend over finite space-time regions. The recourse to the local character of the principle of equivalence used by BONDI and GOLD and adopted in reference ([3]) is therefore not necessary. This is essential, because in homogeneous gravitational fields (which *do* exits over *finite* space-time regions) the principle is no longer a local principle.

RIASSUNTO (*)

Si definisce in termini covarianti secondo Lorentz l'impulso e l'energia della radiazione elettromagnetica emessa da una carica. Come criterio necessario e sufficiente per la misura dell'emissione di una radiazione a un dato tempo si propone la misura non locale di un invariante di Lorentz. La misura consiste di una misura di campo che può essere eseguita a *qualsiasi* distanza dalla carica e non occorre sia fatta nella zona delle onde. Si può considerare come una definizione operazionale della radiazione in termini del tasso di emissione dell'energia. Questi risultati consentono una semplice derivazione dell'equazione del moto di una particella carica che permette di evitare le difficoltà dovute all'autoenergia. Si risolve un quesito concernente il principio d'equivalenza e la radiazione nel moto uniformemente accelerato.

Dispersion Relations and Diffraction Scattering.

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Summary. — An approximate formula for high-energy elastic scattering is derived from dispersion relations.

1. – Introduction.

From dispersion relations, we derive an approximate expression for elastic scattering at high energies. It is of the form of conventional diffraction scattering, with a strong forward peak whose width is related to the range of the interaction. The approximations which we make are crude, but we believe that our formula has the essential features of a more accurate treatment.

2. – High-energy scattering.

Consider the amplitude $T(s, t)$ for the elastic scattering of scalar particles, masses M and μ, where

$$(2.1) \qquad s = E^2 \,,$$

the square of the centre-of-mass energy, and

$$(2.2) \qquad t = -4k^2 \sin^2 \theta/2 = -\Delta^2 \,,$$

the covariant momentum transfer, (k and θ are centre-of-mass momentum and

scattering angle, respectively). We also introduce the variable u so that

(2.3) $$s + t + u = 2M^2 + 2\mu^2 .$$

We normalize T so that the optical theorem takes the form

(2.4) $$\operatorname{Im} T = 2kE\sigma_{\text{Tot}} .$$

Thus in the high energy limit, $(E \gg M)$,

(2.5) $$\operatorname{Im} T = s\sigma_{\text{Tot}} .$$

Then the differential elastic cross-section is

2.6) $$\frac{d\sigma}{d\varDelta^2} = \frac{1}{64\pi E^2 k^2} |T|^2 .$$

To obtain an expression for the imaginary part of T we exploit an idea due to GILBERT [1]. We write a dispersion relation for fixed (small) value of $(= -\varDelta^2)$ for

$$\frac{T^t(s, u)}{(s_c - s)^{\frac{1}{2}}(u_c - u)^{\frac{1}{2}}} .$$

where s_c and u_c are the starting values of the continuum in the appropriate channels. For large positive values of s (and hence large negative values of u) we have

(2.7) $$\frac{\operatorname{Im} T^t(s, u)}{(s - s_c)^{\frac{1}{2}}(u_c - u)^{\frac{1}{2}}} = \frac{1}{\pi} \int_{s_c}^{\infty} \frac{Rl T^t(s')}{(s' - s_c)^{\frac{1}{2}}(u_c - u')^{\frac{1}{2}}(s' - s)} ds' +$$

$$+ \frac{1}{\pi} \int_{u_c}^{\infty} \frac{Rl T^t(u')}{(s_c - s')^{\frac{1}{2}}(u' - u_c)^{\frac{1}{2}}(u' - u)} du' .$$

The main contribution may be expected to come from the neighbourhood o the singularity in the first integral. We have ignored the pole terms (if any), since for $s \gg M^2$ they are never important. In the same spirit we shall also ignore the integral over u'.

[1] W. GILBERT: *Phys. Rev.*, **108**, 1078 (1957).

To determine the real part of T at high energies we use the single variable dispersion relation for fixed s, which we emphasise by writing

$$T(s, t, u) = T^s(t, u) .$$

Then

$$(2.8) \quad RlT^s(t, u) = \frac{g^2}{\mu^2 - t} \mp \frac{g^2}{\mu^2 - u} + \frac{1}{\pi} \int\limits_{t_c}^{\infty} \frac{\mathrm{Im}\, T^s(t')}{t' - t}\, \mathrm{d}t' + \frac{1}{\pi} \int\limits_{u_c}^{\infty} \frac{\mathrm{Im}\, T^s(u')}{u' - u}\, \mathrm{d}u' ,$$

where g is the renormalized coupling constant, and t_c and u_c are the starting values of the continuum mass values (squared) in the appropriate channels. We are interested in evaluating $T^s(t, u)$ for

$$s \gg M^2 ,$$

and for negative values

$$t = - \Delta^2 ,$$

which we show below are small $(\sim - \mu^2)$ for physically important values. Thus

$$(2.9) \qquad u = 2 M^2 + 2\mu^2 + \Delta^2 - s$$

is large and negative and there are no singularities on the right hand side of (2.8). The pole which is most closely approached is that in the first term and in the crudest approximation we can take

$$(2.10) \qquad RlT^s(t, u) = \frac{g^2}{\mu^2 - t} = \frac{g^2}{\mu^2 + \Delta^2} .$$

This is just the « pole » approximation, and we have merely re-derived the well-known result that at high energies and small momentum transfer the real part of the amplitude is dominated by this term.

Substituting (2.10) into (2.7) (with neglect of the integral over u') gives the explicit expression

$$(2.11) \qquad T(s, - \Delta^2) = \frac{g^2}{\Delta^2 + \mu^2} \left(1 + \frac{i}{\pi}\, f(s) \right),$$

where

$$(2.12) \qquad f(s) = (s - s_c)^{\frac{1}{2}} (u_c - u)^{\frac{1}{2}} \int\limits_{s_c}^{\infty} \frac{\mathrm{d}s'}{(s' - s_c)^{\frac{1}{2}} (u_c - u')^{\frac{1}{2}} (s' - s)} .$$

Here u is given by (2.9) and u' is similarly related to s'. This is our main result.

The integral (2.12) may be evaluated explicitly. In the high-energy limit it gives

(2.13)
$$f(s) \simeq \log (4s/s_c) .$$

The high energy behaviour in this approximation is thus:

(2.14)
$$\begin{cases} \operatorname{Im} T \sim \log (4s/s_c) , \\[2ex] RlT \sim \text{const} . \\[2ex] \sigma^{\text{el}} \sim \dfrac{1}{s^2} (\log 4s/s_c)^2 , \\[2ex] \sigma_{\text{Tot}} \sim \dfrac{1}{s} \log (4s/s_c) . \end{cases}$$

The angular distribution is

(2.15)
$$\frac{\mathrm{d}\sigma_{\text{el}}}{\mathrm{d}\varDelta^2} \sim \left(\frac{1}{\varDelta^2 + u^2} \right)^2 ,$$

which is a sharp forward peak with

(2.16)
$$\varDelta^2_{\max} \simeq \mu^2 .$$

This, of course, corresponds to scattering into a forward cone of angle

(2.17)
$$\theta_{\max} \sim \mu/k .$$

This is the usual diffraction formula, corresponding to a potential of range $1/\mu$, with sharp forward peaking, and an amplitude dominated at high energies by the imaginary part.

3. – Applications.

It so happens that for the physically most accessible interactions of pion-pion, pion-nucleon and nucleon-nucleon scattering, the pole term in (2.8) is either absent, or clearly does not dominate the situation, and a better approximation is required. In such cases the largest contribution may be expected to come from the integral over t' in (2.8), which corresponds to the potential between the scattering particles. The value t_c is the least continuum mass

(squared) which can be exchanged by the particles, and determines the maximum range of the interaction. The simplest approximation is to take

(3.1)
$$\text{Im } T^s(t) = \Gamma^2 \, \delta(t - t_R) \qquad\qquad t_R > t_c$$

so that

(3.2)
$$Rl \, T^s(t) \simeq \frac{\Gamma^2}{t_R - t}.$$

The parameters Γ and t_R determine the strength and range of the interaction. It is only for simplicity that we assume that Γ^2 is independent of s. If this restriction were lifted it would alter the high-energy behaviour described in (2.14).

For pion-pion and pion-nucleon scattering this model may be applied, assuming the interaction is mediated by an effective scalar particle. At $E_c \sim 6\,\text{GeV}$ total elastic scattering cross-sections of the order of several millibarns are obtained for interaction ranges corresponding to

$$t_R \sim 4\mu^2$$

and for values of Γ corresponding to effective coupling constants of the order of one. The total cross-section determined via the optical theorem is of the order of about forty millibarns for the same parameters.

In the case of proton-proton scattering there is, of course, a pole term corresponding to exchange of a single meson. However for pseudo-scalar mesons, the sum over spins gives rise to a factor of $\varDelta^2$ in the numerator, so that the matrix element has the form

$$\sum_{\text{spin}} |T|^2 \sim \left(\frac{\varDelta^2 g^2}{\varDelta^2 + \mu^2} \right)^2.$$

This clearly does not dominate the forward scattering. In this case also the main contribution comes from the integrals. The integrand $\text{Im } T^s(t)$ may depend on five form factors whose γ-dependence may be taken to be the standard S.P.V.T.A. interactions conventionally used in β-decay. Taking the scalar type interaction as typical, and again assuming that the corresponding for factor can be approximated by (3.1), the matrix element is now

$$\sum_{\text{spin}} |T|^2 \left(\frac{M^2 \Gamma^2}{\varDelta^2 + t_r} \right)^2,$$

which gives a pronounced forward peak. The backward-forward symmetry of

the angular distribution is ensured by the integral over u' which must give rise to a similar peak in the backward direction. Again an elastic cross-section of several millibarns and a total cross-section of several tens of millibarns is obtained at $E_c \simeq 7$ GeV$/c_c^2$ for reasonable values of the parameters similar to those quoted above. This is in general qualitative agreement with the experimental results [2]

[2] S. G. GOLDSACK, L. RIDDIFORD, B. TELLINI, B. R. FRENCH, W. NEALE, J. R. NORBURY, I. O. SKILLICORN, W. T. DAVIES, M. DERRICK, J. H. MULVEY and D. RADOJICIC: (to be published).

RIASSUNTO [*]

Dalle relazioni di dispersione si deriva una formula approssimata per lo scattering elastico alle alte energie.

Improvements in the Purification of Germanium by Zone Melting.

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(ricevuto il 20 Luglio 1961)

Summary. — To improve the purification of Ge by zone melting a new method of agitation was experienced, based on the dependency of the repulsive forces on the frequency by induction heating. The achieved stirring resulted of a particular efficiency relating to the starting concentration of the material to be purified. The results have been applied on an industrial scale for the reclamation of the scrap germanium.

The efficiency of purification by zone melting is, as known, limited by the velocity of distribution in the melt of the impurities issuing from the solid.

To represent such limitation the following expression, specific for the case of distribution by diffusion, can be used [1]:

$$(1) \qquad v \leqslant \frac{G \cdot \varDelta}{m \cdot C_0} \frac{k}{1-k},$$

where v is the rate of travel of the solidification front, G the thermal gradient on the solidification front, $\varDelta$ the diffusion coefficient of the impurity in the melt, k the distribution coefficient, C_0 the solid concentration on the solidification front, m the slope of the liquidus line.

This limit for the speed of solidification corresponds to the conditions necessary to avoid the constitutional super-cooling [2].

[1] C. Elbaum: *Prog. Met. Phys.*, **8**, 213 (1959); W. A. Tiller, K. A. Jackson, J. W. Rutter and B. Chalmers: *Acta Met.*, **1**, 428 (1953).

[2] J. W. Rutter and B. Chalmers: *Can. Journ. Phys.*, **31**, 15 (1953).

Several attempts ([3]) to increase this limit have been made, all of them essentially based on forced distribution of the impurities by agitation of the liquid.

Here we describe a method of agitation, which resulted particularly efficient and easy.

Fig. 1. – Sketch of the disposition refering to Eq. (2).

To explain this method we will recall the expression of the work power P, absorbed in a solid cylindric induct with an axial flux (Fig. 1) ([4])

$$(2) \qquad P = c \cdot f \cdot H_0^2 \cdot l \cdot \mu \cdot S \cdot Q ,$$

where c is a constant, f is the current frequency, H_0 is the magnetic field near the surface of the work-piece, l is the length of the piece, μ is the coefficient of magnetic permeability, S is the section of the piece, and Q is a function of D/δ ([*]), whose diagram is given in Fig. 2. D being the work-piece thickness and δ the equivalent current depth,

$$(3) \qquad \delta = \frac{1}{2\pi} \sqrt{\frac{\rho}{\mu f}} ,$$

where ρ is the resistivity of the work-piece material and, as before, μ and f are the permeability and the frequency.

Fig. 2. – Values of the Q function plotted against work-diameter to current-depth ratio D/δ.

Now, from the magnetic field H_0 and the induced current I it derives a repulsive force F on the liquid, proportional to $H_0 \cdot I$.

For constant power P, considering I^2/δ as a constant in the induct, we

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([3]) W. G. PFANN: *Zone Melting* (New York, 1958); N. L. PARR: *Zone Refining and Allied Techniques* (London, 1960); W. G. PFANN and D. DORSI: *Rev. Sci. Inst.*, **28**, 720 (1957); B. N. ALEXSANDROV, B. I. VERKIN and B. G. LA'ZAREV: *Fizika Metallov i Metalvedenie*: **2**, 93 (1956).

([4]) P. G. SIMPSON: *Induction Heating* (New York, 1960).

$$(*) \qquad Q = \frac{2}{\xi a} \cdot \frac{\text{bei}' \, \xi a \cdot \text{bei} \, \xi a + \text{ber}' \, \xi a \, \text{ber} \, \xi a}{\text{ber}^2 \, \xi a + \text{bei}^2 \, \xi a}$$

where $\text{ber} \, x + i \, \text{bei} \, x = J_0(ix\sqrt{i})$ is the first kind zero order Bessel's function, $\xi = 8\pi^2 \mu f/\rho$ and a is the radius of the work piece.

have from (2) and (3)

$$F \propto (Q)^{-\frac{1}{2}}(f)^{-\frac{3}{4}} .$$

That is induction heating at lower frequencies will help, as otherwise known, in the agitation of the melt.

Given the rapid decrease of Q it is advisable, on the other side, to work with a frequency corresponding to a D/δ near to the maximum's abscissa, in order to avoid excessive agitation which could disturb the regularity in the advancement of the solidification front.

For $D/\delta = 2.5$ we have

$$f = \frac{\varrho}{\mu}\left(\frac{2.5}{2\pi D}\right)^2 .$$

Replacing the values for the molten germanium

$$\varrho = 6 \cdot 10^4 \, ab \text{ ohm} \cdot \text{cm} \; (^5) \; ; \qquad \mu = 1$$

and for $D = 1$ cm, size order of the diameter of the utilized ingots, we have

$$f = 9.5 \cdot 10^3 \text{ Hz} .$$

Some tests were tried with induction heating at 10 000 Hz and it resulted indeed a very strong agitation of the liquid. So strong that to avoid the
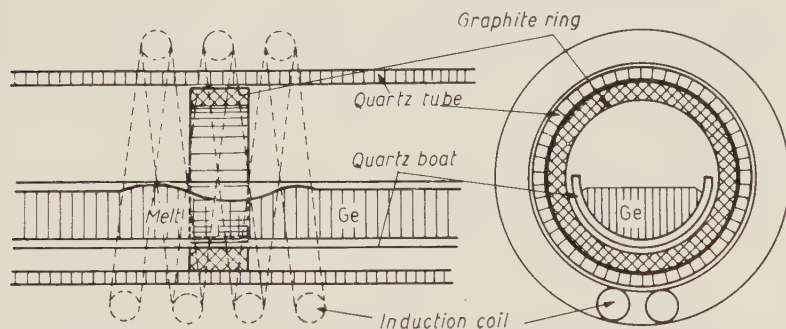
Fig. 3. – Coil design for combined induction and resistance heating for zone melting and stirring.

expulsion of the melt from the boat, as well to preheat the germanium, it has been necessary to use combined induction and resistance heating, according to the sketch of Fig. 3.

(⁵) A. I. Blum, N. P. Mokrovsij and A. R. Regel: Izvestija Akad. Nauk. SSSR, 16, 139 (1952); R. W. Keyes: Phys. Rev., 84, 367 (1951).

In the center of the coil has been placed a graphite ring (H.P.) which is brought to incandescence by induction, helping the heating of germanium and reducing the electromagnetic field necessary for the melting.

Given the limited width of the ring, its shielding could be neglected.

The effects of this agitation have been checked in relation to the velocity of solidification and impurity concentration, the two parameters of main interest for the limit expressed by (1).
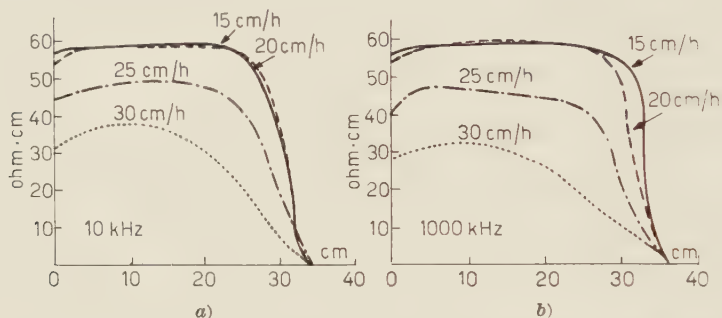
Fig. 4. – Purification by zone melting with induction heating at two different frequencies (10 kHz in Fig. 4a); 1000 kHz in Fig. 4b). Each diagram corresponds to a different speed of solidification, as marked. The purities are expressed by the electrical resistivities, measured along the ingots.

The tests consisted in comparing the degree of purification attained at 10 000 Hz with that obtained by induction heating at $\sim 10^6$ Hz (in the field of the commonly used frequencies, with negligible agitation).
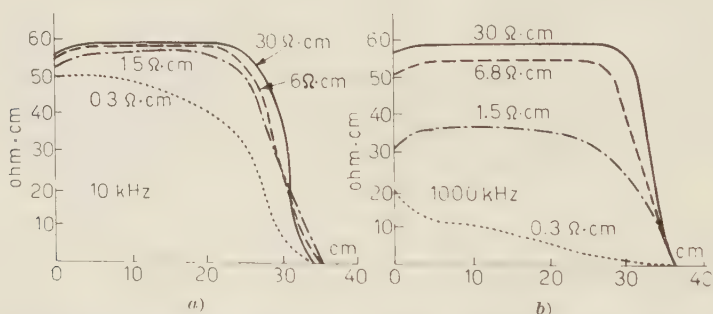
Fig. 5. – Purification by zone melting with induction heating at two different frequencies (10 kHz in Fig. 5a); 1000 kHz in Fig. 5b)). Each diagram corresponds to a different concentration of the starting germanium, as marked.

The results are illustrated in Fig. 4 and 5. The purities are expressed by the resistivities in ohm·cm, measured with the four points method.

As can be seen the effect of the agitation is small with respect to the limit rate of travel; it is on the contrary very remarkable as concerns the concentration of the impurities in the starting material.

We think that the lack of effect in the first case is to be connected with some irregularities in the advancement of the front of solidification, depending perhaps from the agitation itself of the liquid.

The strong increase in purification that we have observed starting from very impure material may be important for the reclamation of scrap germanium.

This result has been already utilized on an industrial scale, as will be described in a next paper ([6]).

([6]) G. SCACCIATI: to be published.

RIASSUNTO

Per migliorare la purificazione del Ge mediante fusione a zone è stato provato un nuovo metodo di agitazione, basato sulla dipendenza delle forze repulsive dalla frequenza nel corso del riscaldamento a induzione. I risultati sono apparsi particolarmente significativi dal punto di vista della concentrazione massima delle impurità nel materiale da purificare. Il procedimento è stato applicato su scala industriale per il recupero del germanio dagli scarti di lavorazione.

Work Hardening and Recovery Phenomena in Aluminium [*].

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Summary. — Structural, mechanical and electrical properties of 99.994% Al sheets have been observed with respect to cold work and recovery degree. According to the percentage reduction by rolling, both mechanical and structural properties develop following three distinct stages. In the first stage the dislocations are spread all over in the crystal; in the second stage there is a development of subgrains, and in the third stage the subgrain size tends to a limit value. During recovery two stages were noted, the first corresponding to subboundaries development, the second to subcrystal growth. The first stage, attributed to dislocation climb, is characterised by an activation energy 1.1 eV. Between subgrain dimension d and hardness H, a dependence $H \propto d^{-0.1}$ is found. From electrical measurements there appeared a contribution to electrical resistivity per unit dislocation of $\sim 4 \cdot 10^{-19}$ ohm·cm.

1. – Introduction.

Recovery phenomena in metals can be analysed on the basis of many properties; mechanical electrical, etc. [1].

These properties present features which are partly independent of each other and the main problem to be solved in the field is still their interpretation on the common basis of their structural characteristics.

At present the techniques of direct observation by the electron microscope give a very good means for studying these structures and the present work is

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[1] P. A. Beck: *Adv. Phys.*, **3**, 245 (1954).

done just on these lines in order to define the correlation between structures and mechanical and electrical properties.

In particular, the subjects refer to:

— The mechanism of subgrain formation. At present dislocation climb is accepted as the ground process [2]; however we have limited informations on whether the climb mechanism occurs either by concentration gradients of vacancies or by internal stresses or else by some other causes.

— Relation between hardness and subcrystalline size, which is still uncertain, as BALL and WARRINGTON [3] have found a quadratic law and BECK et al. [4] a law with a smaller exponent.

Cold work phenomena can be introduced in connection with this last point as they are still rather undefined in relation to subcrystalline features and they will be considered here just in this respect.

With regard to electrical measurements we point out that the results obtained can be inserted in the problem of the contribution to electrical resistivity of the dislocations.

As known the calculations done at first by HUNTER and NABARRO [5] have given very low values; and the following development of KLEMENS, SEEGER and STEHLE [6], HARRISON and HOWIE [7] though nearer, remain still lower than the experimental values [8] so far obtained.

2. – Experimental.

Our observations and measurements were carried out on samples of 99.994% pure Aluminium.

The specimens for observation with electron microscope in transmission, were prepared with four different techniques: by electrolytic polishing with

[2] H. VAN BUEREN: *Imperfections in Crystals* (Amsterdam, 1960).

[3] C. J. BALL: *Phil. Mag.*, **2**, 1011 (1957); *Journ. Iron and Steel Inst.*, **191**, 232 (1959); D. H. WARRINGTON: *Proc. Europ. Reg. Conf. Electron Microscopy* (Amsterdam, 1961); N. J. PETCH: *Journ. Iron and Steel Inst.*, **174**, 25 (1953).

[4] P. A. BECK, B. G. RICKETTS and A. KELLY: *Trans. A.I.M.E.*, **215**, 949 (1959); A. H. LUTTS and. P. A. BECK: *Trans. A.I.M.E.*, **206**, 1226 (1956).

[5] S. C. HUNTER and F. R. N. NABARRO: *Proc. Roy. Soc.*, A **220**, 542 (1953).

[6] A. SEEGER and H. STEHLE: *Zeits. Physik.*, **146**, 242 (1956); P. G. KLEMENS: *Canad Journ. Phys.*, **34**, 1212 (1956).

[7] W. A. HARRISON: *Journ. Phys. Chem. Phys.*, **5**, 44 (1958); A. HOWIE: *Phil. Mag.*, **5**, 251 (1960).

[8] L. M. CLAREBOROUGH, M. E. HARGREAVES and G. W. WEST: *Proc. Roy. Soc.*, A **232**, 255 (1955); *Acta Met.*, **5**, 738 (1957).

Lenoir ([9]) and perchloric acid solution ([10]); by chemical polishing with Alcoa-R5-bright dip ([11]) and with 0.1% HF water solution ([12]).

Polishing with the second and fourth method was reserved to the specimens treated at the lowest temperatures.

The comparison between the images obtained with these different methods of thinning enables us to distinguish surface etch-figures from internal structures, as the first only vary from reactive to reactive.

Some authors ([11-13,14]) have already dealt with this question.

Besides the comparison between the different etch-figures we also used the following technique of direct observation *in situ*. After a preliminary observation the sample is heated by the electron beam until the Aluminium fuses. A thin film of oxide is left, which is a replica of the surface and thus reveals the etch figures (Fig. 1). The metal remains in small spheres over the film.
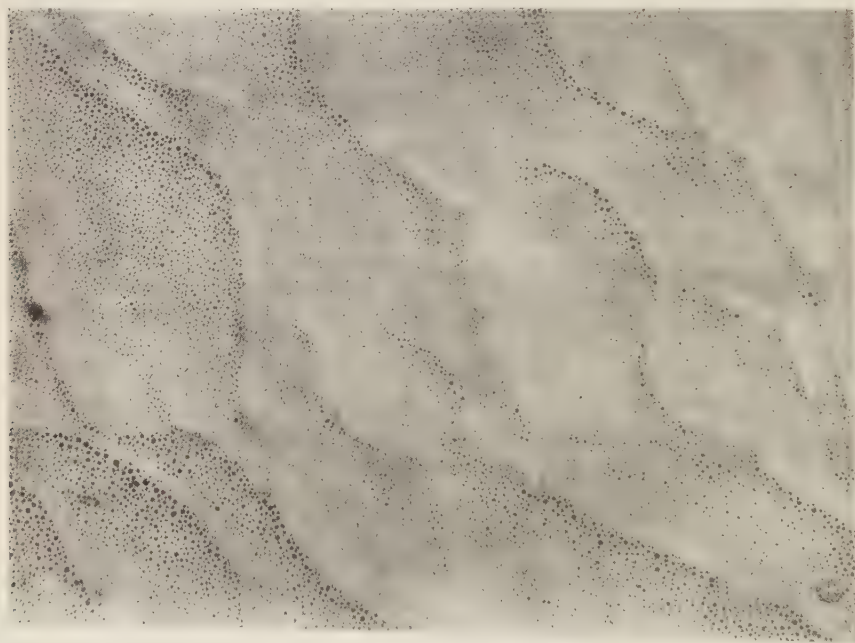
Fig. 1. – Plate I; 50000×. Oxide replica by melting of the Aluminium film.

([9]) G. LENOIR: *Recherche Aéronautique*, **16**, 35 (1953).

([10]) H. H. TOMLINSON: *Phil. Mag.*, **3**, 167 (1958).

([11]) H. BICHSEL: *Metall*, **14**, 196 (1960).

([12]) P. B. HIRSCH, R. W. HORNE and M. J. WHELAN: *Phil. Mag.*, **1**, 677 (1956).

([13]) R. PHILLIPS and N. C. WELSH: *Phil. Mag.*, **3**, 801 (1958).

([14]) H. G. F. WILSDORF: *Symposium in Electron Metallography*, *A.S.T.M.* No. 245 (1958).

The thinning of the specimens was carried out starting from sheets about 20 μm thick.

The thermal treatments, about which we shall speak later, were carried out on sheets with this thickness. The cold work was carried out in such a way to have always 20 μm as the final thickness.

Observations were made with a Siemens-Elmiskop I electron microscope, operating at 100 kV.

In particular, here, we limit ourselves to point out that with the highest cold-worked specimens, it was necessary to use stereoscopic observation to resolve each subcrystal, owing to the low misorientations.

The angles of misorientation between subcrystals were estimated by means of extinction tilting and from the geometry of the dislocations.

Usually the subboundaries crossed the whole thickness of the specimen. The subcrystalline dimensions are given as the square root of the surface of the subcrystals as projected on the photographic plate.

In order to be able to compare hardness measurements with microscopic observations, the former were carried out on sheets whose thickness was comparable to that of the specimens to be thinned for the electron microscope. Their thickness was about 75 μm; it was therefore necessary to use microhardness measurements.

To verify the absence of thickness effects, Meyer diagrams were at first obtained. The thickness effect is shown in deviations from the Meyer line. For example Fig. 2 shows that the thickness effect begins to be felt with diagonals of about 50 μm corresponding to a depth of penetration of ∼ 10 μm.

Usually we worked with diagonals of about one half of this limit.

As it was more convenient and the range of hardness was restricted, constant load hardness (of 17.2 g) was taken instead of constant diagonal hardness.

Also the electrical resistivity measurements were carried out on sheets with thicknesses of the same order of magnitude (40 μm thick, 100 cm long, 3 mm wide).

Fig. 2. – Meyer diagram for microhardness on 75 μm thick Al sheets.

These last measurements were carried out in a boiling Nitrogen bath, by a potentiometric combined volt-ammeter method.

The thermal treatments (both for this last and for the other measurements mentioned above) were done in oil baths.

The passage from the oil bath to the Nitrogen bath took place by means of slow cooling, in order to avoid quenching phenomena.

In order to check the reproducibility, the measurements were repeated three times.

3. – Results.

3'1. *Recovery*. – As already mentioned, recovery tests consisted in following the variations of structural, electrical and mechanical properties with respect to recovery time and temperature.

The specimens were previously cold worked, to the given thickness, by 99% rolling.

Thermal recovery treatments began soon after rolling.

For the measurements of electrical conductivity the time evolution of the phenomena was followed on the same specimens with successive coolings and heatings.
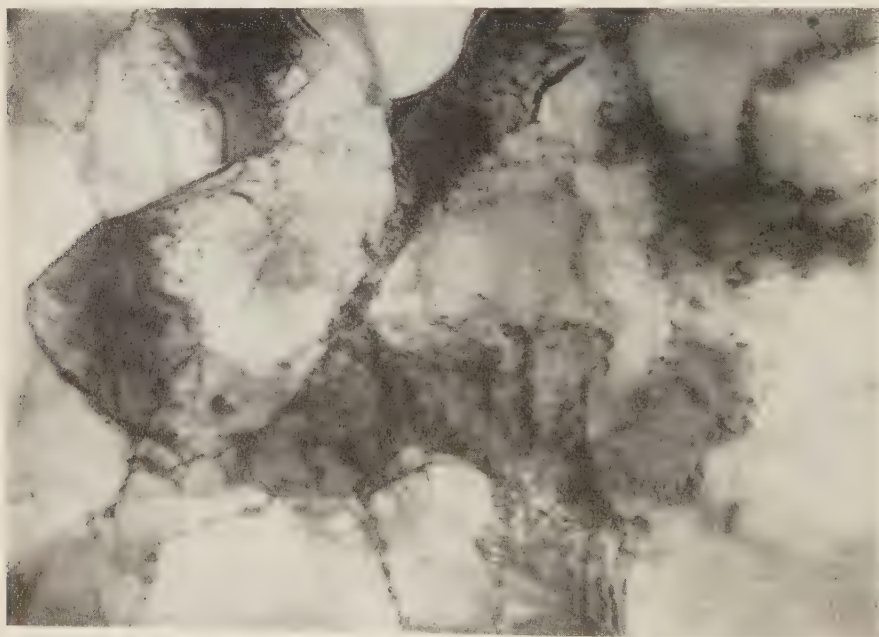
Fig. 3. – Plate II; 25000×. Substructures in just rolled Aluminium.

For the measurements of hardness we used both this method on single specimen and the other one on more specimens, each treated for a different length of time.

For the structural observations only this second method was used.

Inconvenience of the first method is that thermal cycling may change the course of the phenomena; on the other hand for the second we can have errors due to inhomogeneity.

In our case for the single cycle of tests made with both methods no difference resulted within measurement errors.

We consider now the results of the microscopic observations.

Fig. 3 (plate II) represents the situation as after rolling: we have subcrystals with boundaries

Fig. 4. – Plate III; 50000×. Subboundaries in just rolled Aluminium.

made up of ragged dislocations, more or less wide. Plate III (Fig. 4) shows in detail such a subboundary.

The dimensions of these subcrystals were about 0.6 μm.
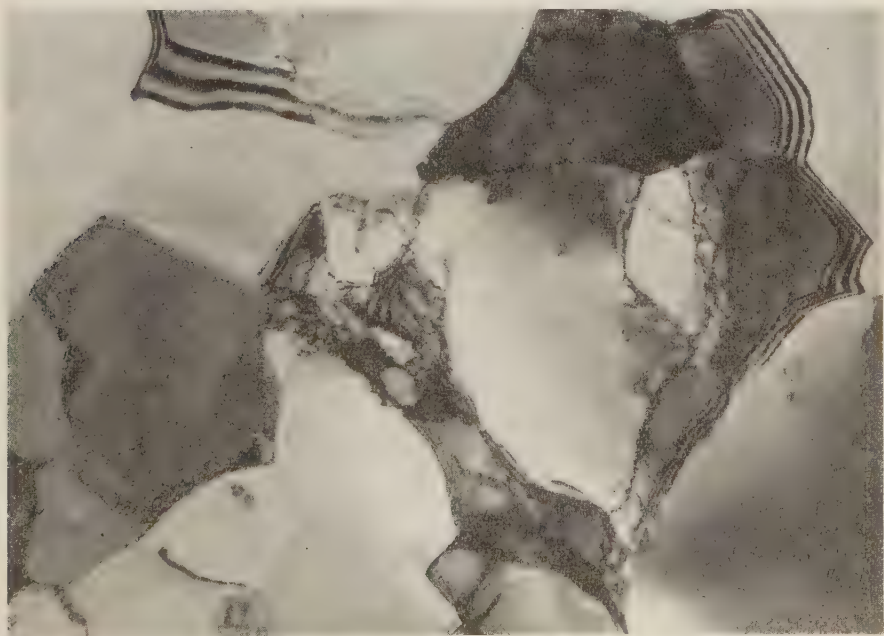
Fig. 5. – Plate IV; 25000×. Substructures in Al heated 30 min at 100 °C.

By heating up to about 100 C there are only small variations in dimensions, while the features of the subboundaries change.

These become sharp, thereby indicating the formation of well defined boundaries (Plate IV (Fig. 5)).

At higher temperatures the mean value and the spread of subcrystal sizes increase, the former up to a limit dimension of about 4 μm.

Plate V (Fig. 6) shows one of these growth structures, for 30 minutes heating at 200 °C.
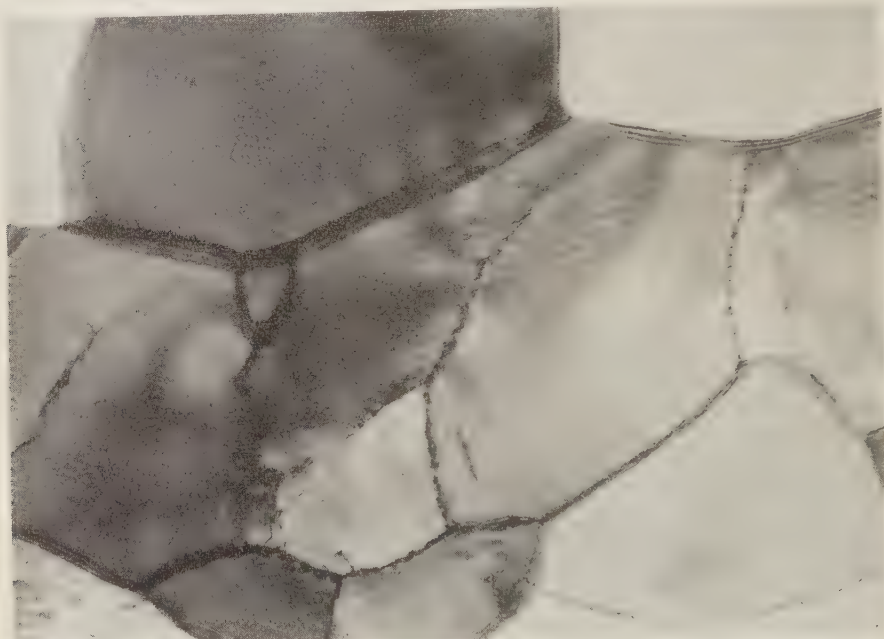
Fig. 6. – Plate V; 25000×. Substructures in Al heated 30 min at 200 °C.

During this stage of subgrowth the angles between adjacent subcrystals show increases from 1°-2° to 5°-10°.

Lastly at 250 °C there is recrystallization (Plate VI (Fig. 7)).

The structural variations with time were not well defined: observations made would indicate a greater dependency on temperature than on time, in the sense that no appreciable growth of the subcrystals was noted in the range of times examined (from 5 minutes to 5 hours).

This phenomenon would agree with a result given by PERRYMAN [15]: but anyway it will be examined more deeply with shorter periods of heating.

[15] E. C. W. PERRYMAN: *Acta Met.*, 2, 26 (1954); *Trans. A.I.M.E.*, 203, 1053 (1955).
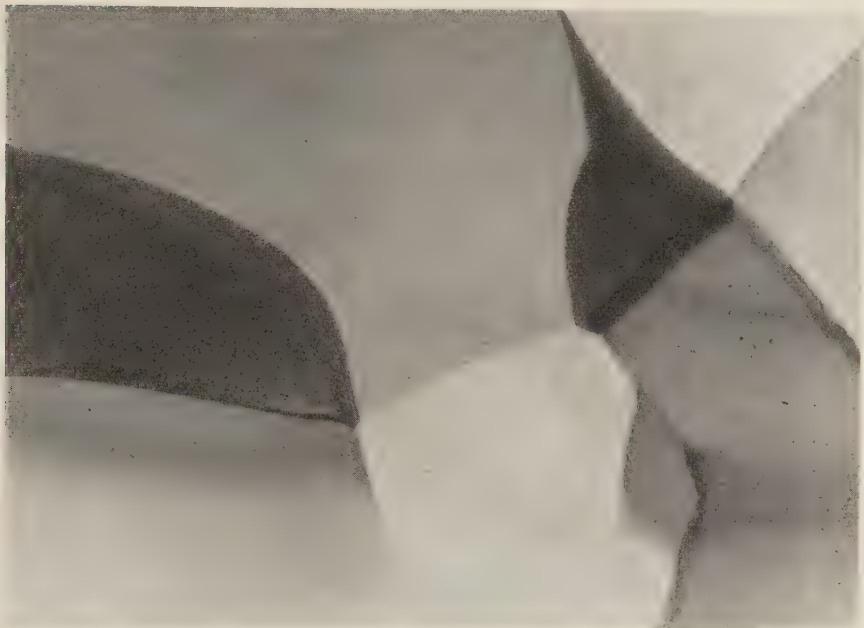
Fig. 7. – Plate VI; 6000×. Recrystallization structure.

As an outline of the observations we give the isochronal of Fig. 8, which represents the mean values of the whole distribution of the subgrain sizes in function of temperature, for heating periods of 30 minutes.

The diagrams (*a*) and (*b*) in Fig. 9 refer also to these observations. They show the order of magnitude of the spread in subcrystalline sizes, for two extreme cases: the smallest spread corresponding to the just rolled material the largest to the beginning of the recrystallization. The tail in the (*b*) distribution being due just to the onset of the recrystallization. The points in Fig. 8 correspond to the arithmetical mean value deduced from such distributions at the different temperatures of recovery.
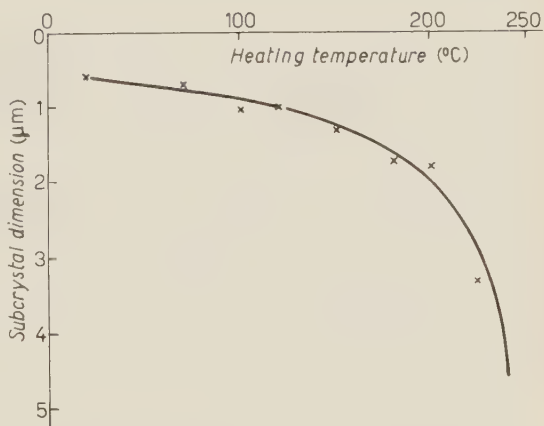
Fig. 8. – Subcrystals dimensions in function of heating temperature. Time of heating 30 min.

Next to indicate the results of mechanical and electrical measurements we

give the diagrams of Fig. 10 and Fig. 11. They are also isochronal, and refer to heating periods of 30 minutes.

The resistivities are given in $\Delta\varrho/\varrho$ %. They are referred to the resistivity of $3.25 \cdot 10^{-7}$ ohm·cm of the just rolled Aluminium at liquid nitrogen temperature.

In comparing the curves in Figs. 8, 10, 11 we see that the electrical resistivity decreases before the other properties.

Fig. 9. – Distribution of Aluminium subcrystal sizes. *a*) Just rolled. *b*) After 30 min recovery at 200 °C. The ordinates are given in percent of the peak value.

With regard to hardness this different behaviour is shown more clearly by the isothermal curves in Figs. 12, 13. From these it is evident that the recovery phenomena connected with the electrical properties evolve more quickly than the others.

We note here how these

Fig. 10. – Microhardness in function of recovery temperature. Time of heating 30 min.

curves correspond with those found by EGGLESTON in Copper ([16]) and WINTER-
BERGER in Aluminium ([17]).
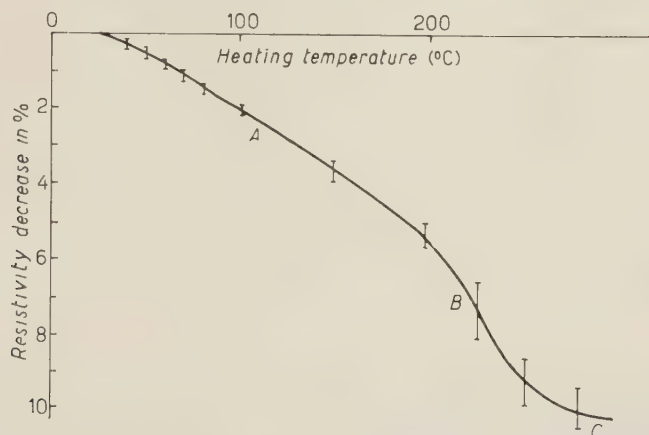
Fig. 11. – Resistivity decreases in function of recovery temperature. Time of heating
30 min. Referring value of resistivity $3.25 \cdot 10^{-7}$ ohm·cm.

From the curves in Fig. 13 an activation energy of (1.1 ± 0.2) eV/mole·°C
can be deduced, with reference to an intermediate degree of recovery. For
this purpose it is better to represent the phenomenon by a logarithmic plot
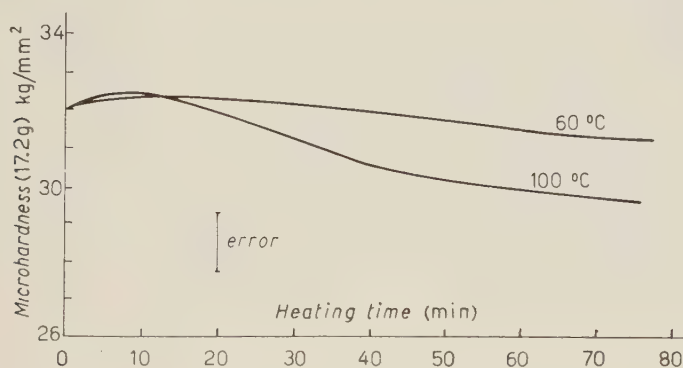
Fig. 12. – Microhardness in function of recovery time for two heating temperatures

(Fig. 13b); the degree of recovery taken as reference is here marked with an
asterisk.

([16]) R. R. EGGLESTON: Journ. Appl. Phys., **23**, 1400 (1952).
([17]) M. M. WINTERBERGER: Compt. Rend., **243**, 128 (1956).

Excluding this initial stage, by comparison between the various curves, an interdipendency between the considered properties can be inferred: that is,
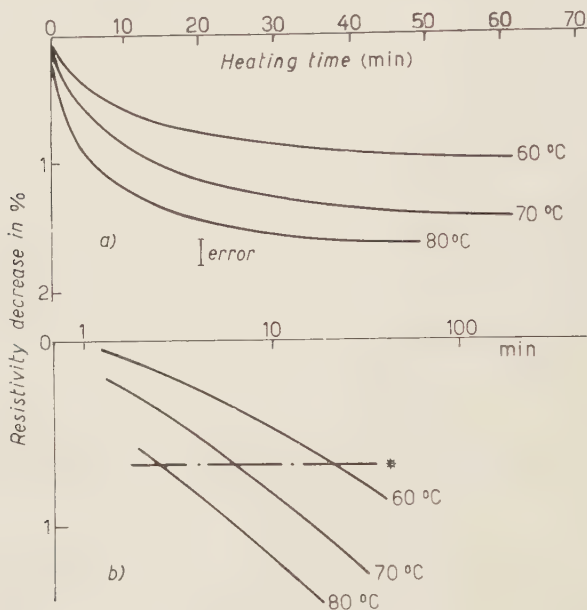
Fig. 13. – Resistivity decreases in function of recovery time for some heating temperatures: *a*) linear; *b*) logaritmic scale.

mechanical and electrical properties should depend on the size of subcrystals. From a plot of hardness *vs.* subcrystalline size a law:

$$H \propto d^{-0,06}$$

can thus be deduced (Fig. 14) as valid within measurement errors.
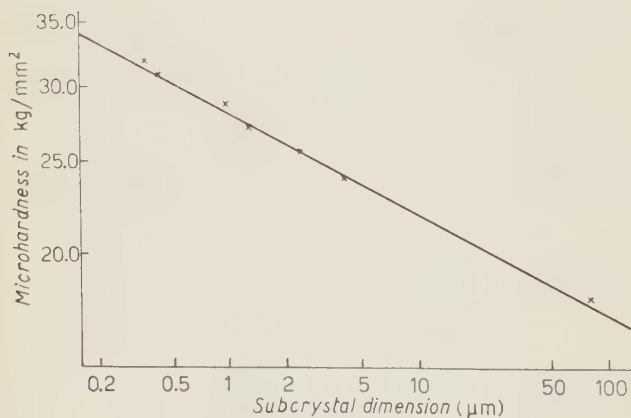
Fig. 14. – Hardness in function of subcrystalline sizes during recovery.

3˙2. *Cold work*. – Referring to this subject we recall first that cold work examinations were made with reference to a final constant thickness of about 20 μm; that is, a series of specimens with various thicknesses was first prepared in order to have the complete range of selected reductions to arrive at this final thickness.

Before cold work the specimens were all recrystallized at 350 °C for an hour, then they were reduced to the final thickness by cold rolling, with one, or at the most two rollings.

From the observations, the following cold work stages resulted.

— With low reduction, up to $\simeq 10\%$, on the inside of the large recrystallized matrix only dislocations randomly distributed are noted, as is shown in Fig. 15 (Plate VII).

Fig. 15. – Plate VII; 20000×. Structures in 8% rolled Aluminium.

— Successively, corresponding to the denser areas of dislocations or in the neighbourhood of grain boundaries, subcrystals are formed, as shown in Fig. 16 (Plate VIII). This situation holds from 10% to 40% ca. of reduction.

— Lastly all the crystals become full of subcrystals, more and more scattered, until, over 90% reduction, subcrystals structure prevails over that of

primitive recrystallization, of which practically no trace can be found. Fig. 3 (Plate II) is near this final situation.
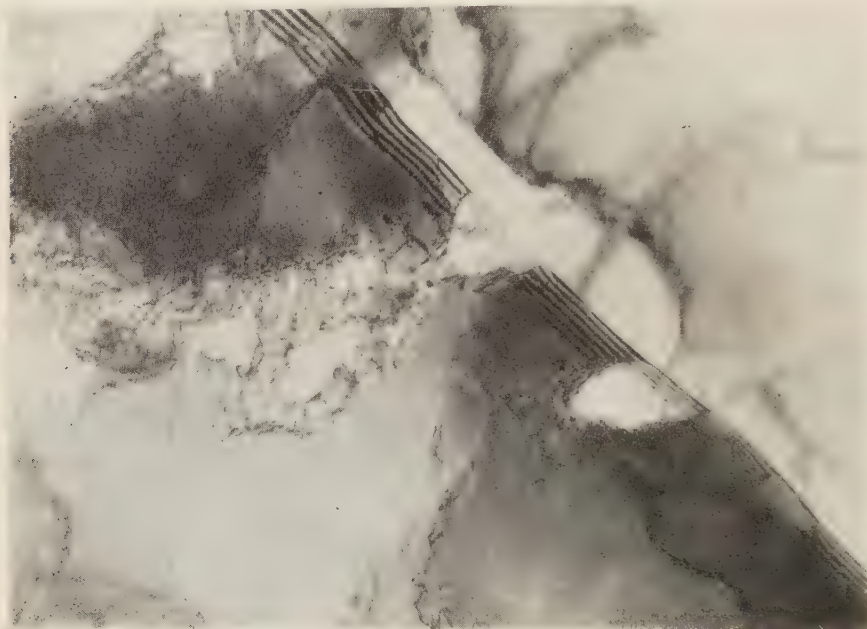

Fig. 16. – Plate VIII; 20000×. Structures in 20% rolled Aluminium.

Also the mechanical properties were followed, by means of the micro-hardness measurements, in function of percentage reductions.

Their behaviour corresponds to that of the structural features as mentioned above, and with an analogous plot as before, we may deduce a correspondence law between hardness and subcrystalline size of the type

$$H \propto d^{-0,1}$$

as shown in Fig. 17.
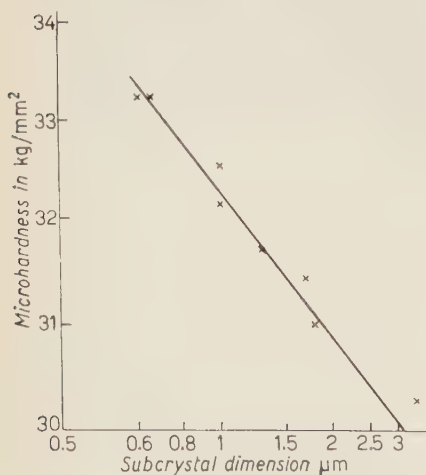
Fig. 17. – Hardness in function of sub-crystalline sizes, by reduction with rolling.

4. – Discussion.

The discussion of the experimental results is based on the following points:

a) The initial decrease of the electrical resistivity and the connected structural changes.

b) The relation between structural and mechanical properties.

c) The contribution of dislocations to the electrical resistivity.

a) Microscope observations have shown that at the same time with the first reduction of electrical resistivity (preceding the growth of subcrystals and the diminution of hardness) we have rearrangement of dislocations in the sub-boundaries.

The two phenomena are therefore considered to depend on each other.

The rearrangement of dislocations should contribute to the resistivity either through annihilation of dislocations or through variations of the concentrations of point defects, particularly vacancies.

In each case the fundamental process should be the climb of dislocations.

The activation energy deduced from the resistivity changes should then be related to this climb.

Now we can consider as more probable the two following mechanisms of climb:

— by internal stresses;

— by forces connected with gradients in the concentration of vacancies.

In the first case the following expression can be considered valid ([18]) for the activation energy Q:

$$(1) \qquad Q \simeq U_f + U_m - \sigma_0 b^3 ,$$

where U_f is the formation energy of vacancies, U_m is the activation energy relative to vacancy migration, σ_0 is the internal stress, b the Burgers' vector.

In the second case we have instead

$$(2) \qquad Q \simeq U_m .$$

We neglect the jog formation energy owing to the high degree of cold work.

[18] J. FRIEDEL: *Les Dislocations* (Paris, 1956).

3415

In Aluminium the migration energy U_m is $\simeq 0.6$ eV. The sum of the two energies, of migration and formation, $U_f + U_m$ is ca. 1.4 eV [19].

Our experimental value would therefore favour the former of the two mechanisms considered.

The behaviour of resistivity with time agrees with such a mechanism: a recovery rate which decreases inversely proportional to time can indeed be connected [18] with an activation energy which decreases linearly with cold work, corresponding to (1).

Perhaps the greater scatters we found in the measurements of hardness in the conditions under consideration (Fig. 10) may be attributed to these internal stresses.

We may check if there are admissible, for $\sigma_0 b^3$, contributions to the activation energy of the necessary order of magnitude.

This contribution must be $\simeq 0.3$ eV; corresponding, for $b = 2.8 \cdot 10^{-8}$ cm, to $\sigma_0 = 2 \cdot 10^{10}$ dyne cm^{-2}.

The number of dislocations in a pile up will be

$$n = \frac{\sigma_0}{\sigma},$$

where σ is the external applied stress.

We must ascertain if they can be piled up inside the subcrystalline dimensions d.

One has

$$(3) \qquad\qquad n = \frac{\sigma_0}{\sigma} \simeq \frac{\pi \cdot d \cdot \sigma}{G \cdot b}.$$

the shear modulus G being equal to $3 \cdot 10^{11}$ dyne cm^{-2}. Solving for σ, we obtain $\sigma = 8 \cdot 10^8$ dyne cm^{-2} with good experimental agreement.

The value we found agrees with the activation energy reported by LUTTS and BECK [1], however in a successive publication BECK, RICKETTS and KELLY [1] found a higher activation energy of $\simeq 1.8$ eV. We think that these different values are connected with the dependency of the activation energy on the degree of recovery. At first, there are high internal strains, then as recovery proceeds, the strains get smaller and Q tends towards $U_f + U_m$; the last increase of Q is probably connected to the thermal equilibrium of the

[19] F. J. BRADSHAW and S. PEARSON: Phil. Mag., 2, 570 (1957); J. TAKAMURA: Metal Phys., 2, 112 (1956); A. S. NOWICK: Journ. Appl. Phys., 22, 1182 (1951); T. FEDERIGHI: Phil. Mag., 3, 1223 (1958).

jogs, with the result that $Q = U + U_m + U_g$, where U_g is the formation energy of these jogs.

KUHLMANN, MASING and RAFFELSPIELER ([20]) found a similar Q variation.

b) The relation between hardness and crystal size we found, will be discussed with particular attention to the law:

$$(4a) \qquad\qquad H \propto d^{-\frac{1}{2}}$$

considered by various authors (BALL, WARRINGTON and PETCH ([3])). According to these authors the effect of the subcrystalline dimensions on the stress-strain characteristics can be explained by identifying such dimensions with the lengths necessary for the accumulation of dislocations in a number n, such as to allow the load concentration $n \cdot \sigma = \sigma_0$ necessary either to overcome the barriers presented by the subboundaries to the movement of these dislocations or for the production of microcracks.

Recalling the above mentioned formula (3) we have, in fact, for σ the following expression

$$(4b) \qquad\qquad \sigma = \left(\frac{G \cdot b}{\pi}\right)^{\frac{1}{2}} \cdot d^{\frac{1}{2}} \cdot \sigma_0^{\frac{1}{2}},$$

which corresponds to (4a) if σ_0 is considered to be independent of d.

In our case the hardness properties we examined may depend both on shear and fracture phenomena.

From the point of view of shear an explanation may result from arguing the constancy of σ_0.

It appears logical to admit that σ_0 depends on the misorientation δ between the subcrystals.

As for the case of pure tilt or twist boundaries, we will consider σ_0 just proportional to ϑ ([21]):

$$(5) \qquad\qquad \sigma_0 = A \cdot \vartheta .$$

On the other hand for ϑ can be given the following approximated relation

$$(6) \qquad\qquad \vartheta \simeq \frac{N \cdot d \cdot b \cdot l}{3},$$

([20]) D. KUHLMANN, G. MASING and J. RAFFELSPIELER: *Zeits. Metallkunde*, **40**. 241 (1949).

([21]) F. R. N. NABARRO: *Advan. Phys.*, **1**, 269 (1952).

where N is the volume density of dislocations and l is the length of the dislocations which will be taken equal to unity, referring to an unitary volume.

Substituting (5) and (6) in (7) we have

$$\text{(7)} \qquad \sigma = \left(\frac{G \cdot b^2 \cdot A}{3\pi}\right)^{\frac{1}{2}} \cdot N^{\frac{1}{2}},$$

that is σ should depend on d only through N.

In particular to explain the law which we found, a relation should be valid of the type

$$\text{(8)} \qquad N = C \cdot d^{-0,2},$$

which could correspond to dislocation annihilation during recovery or to dislocation multiplication during cold work.

We can see if the amount of annihilation, as indicated by (8), agrees with the observed variations of ϑ.

Our observations have given, for d varying from 10^{-4} to $2.5 \cdot 10^{-4}$ cm, a ϑ increase from ca. 0.07 to 0.12 rad.

Using (6) this corresponds to

$$\Delta N = \Delta \left(\frac{3\vartheta}{d \cdot b}\right) \simeq 2 \cdot 10^{10} \, \text{disl} \cdot \text{cm}^{-3}.$$

On the other hand from (8) and for d and ϑ varying in the same range we obtain

$$\Delta N = C \cdot \Delta(d^{-0,2}) \simeq 1 \cdot 10^{10} \, \text{disl} \cdot \text{cm}^{-3},$$

which is of the same order of magnitude.

As we will see later also the decrease of electrical resistivity seems to agree with the hypothesis above.

However the dependency of σ_0 on ϑ, though referring not to the same conditions, seems to contradict some experimental results of BALL [3] and WARRINGTON [22], which lead to the $d^{-0,5}$ law.

A deeper insight in the question will be necessary, from the experimental and theoretical point of view, taking into account the crossing of dislocation forest [23] and the mechanism of activation of dislocation sources at boundaries [22,24].

[22] D. H. WARRINGTON: Dissertation (Cambridge, 1960).

[23] Z. S. BASINSKI: Phil. Mag., 4, 393 (1959); P. B. HIRSCH: Met. Rev., 4, 101 (1959).

[24] J. FRIEDEL: Simpos. on Internal Stresses and Fatigue in Metals, Ed. by G. M. RASSWEILER and L. GRUBE (London, 1958).

Going on to fracture phenomena we note that the subcrystalline dimensions, which we give, refer to predeformation conditions, while fracture depends on the final ones, preceding the fracture itself: so the initial dimensions are to be taken into account only for the influence they have on these final dimensions.

Now the final situation can be suggested by the observations of HAASE and GRANSER ([25]); these authors have carried out observations with the electron microscope in correspondence to microhardness impressions, noting a banded structure according to the penetrator geometry, therefore in first approximation independent of the metal structure.

Probably fracture depends on the distribution of dislocations in these bands, thereafter the subcrystalline dimensions would have only a limited secondary effect on the fracture itself, according to the experimental law which we found.

c) As calculated above, from the beginning to the end of subcrystal growth (point A to B in Fig. 11) we may compute a dislocation decrease $\Delta N \simeq 2 \cdot 10^{10}$ disl·cm^{-3}.

Comparing this with the corresponding resistivity decrease ($\Delta \varrho_{AB} = 13 \cdot 10^{-9}$ ohm·cm) we deduce a contribution to the electrical resistivity, pro unitary dislocation, of $6 \cdot 10^{-19}$ ohm·cm.

We may try to obtain an analogous expression for the transition to recrystallisation (from B to C in Fig. 11).

Using (6) and neglecting the dislocation density after recrystallization we have

$$\Delta N = \frac{3 \vartheta_B}{b \cdot \bar{a}_B} = 5 \cdot 10^{10} .$$

The corresponding $\Delta \varrho_{AB}$ is $11 \cdot 10^{-9}$ ohm·cm, thus indicating a mean dislocation contribution of $2.2 \cdot 10^{-19}$ ohm·cm in agreement with the other above.

These results, obtained with dislocations grouped in subboundaries, can be considered in a general sense only if short range scattering prevails in resistivity. This agrees with theoretical assumption (5).

The values we found correspond to the one obtained in Copper by CLAREBOROUGH ([8]) and has been approximated theoretically by estimates made recently by HARRISON and HOWIE ([7]), though remaining greater by an order of magnitude.

Probably this discrepancy is due to vacancies which are still to be present.

([25]) G. HAASE and F. GRANSER: *Naturwiss.*, **47**, 225 (1960).

5. – Conclusion.

Recovery and work-hardening phenomena were followed on 99.994 % Alu minium sheets by direct electron microscopy, electrical resistivity and hardness measurements.

Referring to work-hardening, by means of rolling at room temperature, three stages were noted, starting from completely recrystallized material.

— Up to 10 % reduction random distribution of dislocations within the crystals.

— Up to 40 % reduction formation of subcrystals with dimensions greatly dependent on the reduction degree.

— Lastly one has a saturation of subcrystal dimensions, towards a limit of the order of magnitude of 10^{-5} cm.

The hardness H follows the structural variations, depending on the dimensions d of the subcrystals according to a law

$$\bar{H} \propto d^{-0.1} .$$

For recovery, the following behaviour was found, starting from 99 % cold-worked sheets.

At first there is a climb of dislocations in subboundaries, without any considerable variation of subcrystalline dimensions. This stage is accompanied by decreases of electrical resistivity, while the mechanical properties remain unchanged. The dislocation climb appears to be controlled by internal tensions.

Later, we have growth of the subcrystals accompanied by softening. The hardness may be connected to the dimensions of subcrystals by a relation similar to the one found for hardening:

$$H \propto d^{-0.06} .$$

Comparing structural and electrical properties a contribution of $\simeq 4 \cdot 10^{-19}$ ohm·cm to electrical resistivity, pro unit dislocation, can be deduced.

It has been attempted an explanation of the relation between hardness and subgrain dimensions based on a dependency of the obstacle action, to dislocation motion, from subboundary angle.

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RIASSUNTO

Su laminati di Alluminio 99.994% sono state osservate le proprietà strutturali, meccaniche ed elettriche in funzione dell'incrudimento e del rinvenimento. In dipendenza del grado di incrudimento, per laminazione, proprietà strutturali e meccaniche seguono tre stadi distinti. Nel primo le dislocazioni appaiono disperse pressochè uniformemente all'interno dei cristalli; nel secondo si ha lo sviluppo progressivo dei subgrani e nel terzo si ha tendenza verso valori limite delle dimensioni dei subgrani stessi. Durante il rinvenimento sono stati osservati due stadi, il primo corrispondente allo sviluppo dei subcontorni, il secondo a crescita dei subcristalli. Il primo stadio, connesso con la diffusione normale delle dislocazioni, è caratterizzato da una energia di attivazione di 1.1 eV. Sia nel corso degli incrudimenti che in quello dei rinvenimenti durezze e dimensioni subcristalline sono apparse correlate da una legge del tipo $H \propto d^{-0,1}$. Le misure del contributo alla resistività elettrica hanno fornito per dislocazione unitaria valori di $\sim 4 \cdot 10^{-19} \ \Omega$ cm.

Hard-Core Effects on the Energy Gap in Nuclear Matter.

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Summary. — It is shown that the energy gap Δ in the spectrum of the free quasi-particles in nuclear matter strongly depends on the relativistic details of the interaction.

In a previous paper [1] (hereafter referred as I) the problem of the existence of an energy gap Δ in the spectrum of the free quasi-particles in nuclear matter was treated. This problem has been studied because a possible way of eliminating the difficulty of the conventional perturbation expansion arising from the pole of the BRUECKNER t matrix [2,4] is to make first a Bogoliubov-Valatin canonical transformation, and then a perturbation expansion [5,6].

But as will be seen later, the calculation of Δ can be important also in the clarification of the concept of the internucleon potential.

In this work we have developed the study of the dependence of Δ on the hard core of the potential.

In (I) the equation for the gap (we will use systematically the same nota-

[1] G. Fano and A. Tomasini: *Nuovo Cimento*, **18**, 1247 (1960).

[2] L. Van Hove: *Physica*, **25**, 849 (1959).

[3] V. J. Emery: *Nucl. Phys.*, **12**, 69 (1958).

[4] C. Bloch: *Comptes Rendus du Congrès International de Physique Nucléaire,* 1958 (Paris, 1959).

[5] G. Fano: *Nuovo Cimento*, **15**, 959 (1960).

[6] A. Tomasini: *Nuovo Cimento*, **20**, 963 (1961).

tions as in (I))

(1)
$$\Delta_k = \int_0^\infty dk' \cdot k'^2 K(k, k') \frac{\Delta_{k'}}{\sqrt{\xi_{k'}^2 + \Delta_{k'}^2}} \,,$$

has been approximated with a linear integral equation (see eq. (25) of (I)):

(2)
$$f(k) = K(k, 1) + \int_{\text{out}} dk' \cdot k'^2 \frac{K(k, k')}{|\xi_{k'}|} f(k') \,,$$

and this equation has been solved by a Neumann series, assuming a potential of the kind (P_M is the Maiorana exchange operator)

(3)
$$\left\{ \begin{array}{l} W = V(r) \left[\dfrac{1 + \beta}{2} + \dfrac{1 - \beta}{2} P_M \right], \\[2mm] V(r) = V_1(r) + V_2(r) \,, \end{array} \right.$$

where

(4)
$$\left\{ \begin{array}{l} V_1(r) = \left\{ \begin{array}{ll} V_1 \,, & \text{for } r < r_c, \\[1mm] 0 \,, & \text{for } r > r_c, \end{array} \right. \\[4mm] V_2(r) = - V_2 \exp[- \alpha r]. \end{array} \right.$$

This potential has the peculiarity that the value V_1 inside the core is finite.

It was found in (I) that a sufficient condition for the convergence of the Neumann expansion is

$$V_1 < 729 \text{ MeV} \qquad \text{for} \qquad \beta = 0 \,,$$

$$V_1 < 811 \text{ MeV} \qquad \text{for} \qquad \beta = -0.4 \,.$$

Therefore for numerical estimates the Neumann expansion is not well suited and the best way to proceed is to reduce the integral equation to a set of n linear equations in n unknowns. Also the integral eq. (2) is not adapted to numerical calculations, since under the integration we have a function having no poles, but which becomes very large near k_F. (See eq. (19) of (I) for the definition of the symbol $\int_{\text{out}}$.) For the purpose of numerical computations, it is desirable to have a more flat function. In order to obtain a function of this kind, we use the following device: from eq. (2) we can write:

$$f(k) = K(k, 1) + \int_{\text{out}} dk' k'^2 \frac{K(k, k') - K(k, 1)}{|\xi_{k'}|} f(k') + K(k, 1) \int_{\text{out}} dk' \frac{k'^2}{|\xi_{k'}|} f(k') \,.$$

Therefore, putting:

$$(5) \qquad a = 1 + \int\limits_{\text{out}} dk' \, \frac{k'^2}{|\xi_{k'}|} \, f(k') \, ,$$

$$(6) \qquad f(k) = au(k) \, ,$$

we have the following equation for $u(k)$:

$$(7) \qquad u(k) = K(k, 1) + \int\limits_{\text{out}} dk' \, k'^2 \, \frac{K(k, k') - K(k, 1)}{|\xi_{k'}|} \, u(k') \, .$$

Obviously, the function under the integration does not become very large for k' near k_F. From eq. (5), (6), one has:

$$(8) \qquad f(k) = \frac{u(k)}{1 - \int\limits_{\text{out}} dk' \, k'^2 \, (u(k')/|\xi_{k'}|)} \, .$$

Therefore in order to obtain the energy gap, it is necessary to get the function $u(k)$ from the eq. (7).

We have reduced the integral equations (7) to a set of ninty equations in ninty unknowns; making use of an electronic computer I.B.M. 704 we have solved this set of equations for various choices of the parameters V_1 and k_F. For any of such choices we have obtained from eq. (26) of (I) and from eq. (8) the corresponding value of the gap Δ.

The results are given in the following table: (in MeV)

	$V_1 = 500$ MeV	$V_1 = 800$ MeV	$V_1 = 1100$ MeV
$k_F = 1.48 \cdot 10^{13}$ cm^{-1}	$0.503 \cdot 10^{-2}$	$0.229 \cdot 10^{-4}$	$0.19 \ \cdot 10^{-8}$
$k_F = 1.44 \cdot 10^{13}$ cm^{-1}	$0.54 \ \cdot 10^{-2}$	$0.338 \cdot 10^{-4}$	$0.102 \cdot 10^{-7}$
$k_F = 1.36 \cdot 10^{13}$ cm^{-1}	$0.619 \cdot 10^{-2}$	$0.853 \cdot 10^{-4}$	$0.159 \cdot 10^{-6}$

Δ results strongly dependent on V_1, also in the region where V_1 is of the order or greater than the mass of the nucleon. This result does not give troubles

as long as we consider the procedure described in (I) (canonical transformation and perturbation expansion) as a trick which eliminates the difficulties of the perturbation theory: for a given V_1 we obtain a Δ which eliminates the difficulties, and which has no physical meaning. But if we consider Δ as a physical quantity, in the sense that we expect that the true gap in the energy spectrum of nuclear matter is of the order of Δ, a big difficulty arises (*).

In fact one can represent the interaction between the nucleons non relativistically, *i.e.* in terms of static potentials, only if such potentials are much smaller than the mass of the nucleon. Therefore if the physical quantities depend on V_1 when V_1 is of the order of the mass of the nucleon, the relativistic details of the interaction are important in the determination of the properties of nuclear matter, and we cannot use static potentials. This case seems rather unlikely; in fact the mean kinetic energy of the nucleons inside the nucleus is much smaller than the mass of the nucleon and it appears reasonable to assume that all physical quantities do not depend on V_1 when it becomes of the order of $(500 \div 1000)$ MeV.

In order to obtain the true energy gap one can follow the procedure previously outlined (6) with a perturbation method, starting from the unperturbed value Δ.

But in presence of forces with a hard core, it is better to eliminate the influence of the hard core from the very beginning. As is well known the way of making this elimination in the theory of the binding energy of the nuclear matter is offered by the use of the Brueckner t matrix.

BRUECKNER *et al.* (7) have used a method in which, in place of the true Hamiltonian, a model Hamiltonian is used which contains the t matrix in place of the potential. Their method contains some approximations which appear to be not important in the determination of the binding energy of the nuclear matter; however these approximations could be of much more importance in the case of the pairing effect, as the gap is given by the famous formula $\exp[-x]$ where x is related to the kernel used in an equation similar to eq. (1) with the t matrix in place of the potential V.

A better procedure seems to be the solution of the so-called « improved gap equation » (8,9) in which one can introduce the t matrix in a more consistent way.

This point will be discussed in a future paper.

(*) This difficulty has been pointed out by Prof. B. FERRETTI.
(7) K. A. BRUECKNER, T. SODA, P. W. ANDERSON and P. MORREL: *Phys. Rev.*, **118**, 1442 (1960).
(8) J. B. SCHRIEFFER: *Physica*, **26**, 124 (1960).
(9) S. T. BELIAEV: *Physica*, **26**, 181 (1960).

* * *

RIASSUNTO

Si mostra che il « gap » Δ nello spettro energetico delle quasi-particelle libere nella materia nucleare dipende fortemente dai dettagli relativistici dell'interazione.

LETTERE ALLA REDAZIONE

Interaction of Protons from Brookhaven Alternate Gradient-Syncrotron with Nuclear Emulsion.

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We have observed 872 nuclear stars produced in Ilford K-5 emulsion of dimensions 10×10 cm and 600 μm thick exposed to the proton beam of the new Brookhaven AGS machine. The stack was exposed in diffracted proton beam. It was about 100 feet downstream from the target at an angle of 1° to the incident proton beam. There was no magnetic analysis. The beam contains thus diffracted protons as well as the secondaries. The average momentum of the beam was 28 GeV/c and the average track density in the plane perpendicular to the beam entering the emulsion was $4.3 \cdot 10^4$ tracks per cm². Individual primary tracks were followed in a total length of 329.3 meters and the mean free path for the star production was $\lambda = (37.9 \pm 1.2)$ cm. The value of mean free path within the limits of experimental errors is approximately the same as obtained at 6.3 GeV [1-3]

and 9 GeV [4] protons as well as at 100 GeV [5] π-mesons.

While scanning along the track for the first 400 stars, we made measurements on all the shower particles ($g \leqslant 1.4\ g_m$). of all types of stars. For the rest of the scanning along the tracks, we only recorded the total track length followed of all individual tracks and the interactions of those interacting with emulsion nuclei, but took measurements on all the shower particles of only «white stars» with N_h (number of grey and black prongs) $= 0$. The events with one secondary track of a white star and with no visible excitation of energy were considered due to «scattering» and were not counted as stars.

As there is some contamination of secondary particles in the primary proton beam, we determined the energy of all the primary interacting tracks with $n_s > 2$ from the angular distribution of

[1] P. J. Jain and H. C. Glahe (unpublished results).

[2] R. M. Kalbach, J. J. Lord and C. H. Tsao: *Phys. Rev.*, **113**, 330 (1959).

[3] H. Winzeler, B. Klaiber, W. Koch, M. Nikolić and M. Schneeberger: *Nuovo Cimento*, **17**, 8 (1960).

[4] N. P. Bogachev, S. A. Bunjatov, T. P. Merekov and V. M. Sidorov: *Dokl. Akad. Nauk USSR*, **121**, 615 (1958).

[5] A. G. Barkow, B. Chamany, D. M. Haskin, P. L. Jain, H. Lohrmann, M. W. Teucher and M. Schein: *Phys. Rev.*, **122**, 617 (1961).

their shower particles by the relation [5,6]

$$\log \gamma_c = - \langle \log \operatorname{tg} \psi_i \rangle \,,$$
$$E_p = n(2\gamma_c^2 - 1) Mc^2 \,,$$

where

$$n = 0.75 \text{ for events with } N_h \leqslant 5 \,,$$
$$n = 1.8 \quad \text{for events with } N_h > 5 \,,$$

γ_c is the energy of the primary particle in the CMS in units of its rest energy; E_p is the energy of the primary particle

particles which made angle $\psi_i < 2^\circ$ with the direction of the primary, considering that these particles are the incoming particles which continue without appreciable deflection and without strongly participating in the collisions with the emulsion nuclei. In the case of events with $N_h = 0$ and 1, when plotting γ_c against the number of events, a very sharp peak for a γ_c value lying between 4.5 and 5 was observed. This corresponds to primary protons of average energy about 28 GeV. There was a spread in

Fig. 1. – Frequency distribution of the number of shower particles for stars with $N_h = 0$ and 1. $2 \leqslant N_h \leqslant 7$ and $N_h > 7$. The average number of shower particles is 4.1, 4.5 and 5.4 respectively. ——— $N_h = 0, 1$; ---- $2 \leqslant N_h \leqslant 7$; .... $N_h > 7$.

per nucleon, and Mc^2 is the rest energy of a nucleon. In making use of the above equations we neglected the shower

(⁶) C. CASTAGNOLI, G. CORTINI, C. FRANZINETTI, A. MANFREDINI and D. MORENO: *Nuovo Cimento*, **10**, 1539 (1953).

the value of γ_c around the peak value. If we consider a symmetrical distribution of events with different γ_c values around the peak value of γ_c, i.e. roughly the same distribution for lower γ_c value as for higher γ_c value, then we find an excess of 8 to 10% of events on the

lower side of γ_c lying between $\gamma_c = 2$ and 3, which may be due to the contamination of secondary particles of lower energy in the primary beam. γ_c values for all events with $N_h > 5$ were also

the peak value, we observed an excess of 10 to 12% of events on the lower side of the γ_c value, lying between $\gamma_c = 1$ and 2, indicating that the contamination due to secondary low energy particles in

Fig. 2. – The angular distribution of charged shower particles in the laboratory system for stars $N_h = 0$ and 1, $2 \leqslant N_h \leqslant 7$ and $N_h > 7$. The theoretical median angle for the $N_h = 0$ and 1 case is about 13°. ——— $N_h = 0, 1$; –––– $2 \leqslant N_h \leqslant 7$; ········· $N_h > 7$.

calculated and when plotted against the number of events, they showed a peak value for γ_c between 2. and 3.5 If we again consider an approximate symmetrical distribution of events about

the primary beam is about 10 to 12%. Thus the overall contamination due to the presence of secondary particles among the primary proton beam is about 10%. The average values of γ_c are equal to

4.3 and 2.8 for events with $N_h \leqslant 1$ and $N_h > 5$ respectively, which corresponds to an average primary energy of about 25 GeV, and this further proves that the contamination of the secondaries is small in the primary beam. In general, it is difficult to identify the individual, secondary contaminating (interacting and non interacting) tracks from the primary beam at these high energies and since the contamination is small, so in the following discussion, we shall disregard the presence of secondaries among the primary beam.

In Fig. 1 are given the frequency distributions of the number of shower particles per star for stars with $N = 0$ and 1; $2 < N_h \leqslant 7$ and $N_h > 7$. The average number of shower particles n_s is 4.1, 4.5 and 5.4 respectively. The angular distributions of the shower particles in the laboratory system, for stars with $N_h = 0$ and 1; $2 \leqslant N_h \leqslant 7$ and $N_h > 7$ are shown in Fig. 2. The increase in n_s with N_h in Fig. 1 and the broadening of the spread in the angular distribution with N_h in Fig. 2 is presumably due to secondary interactions of the shower particles with the parent nucleus.

There was a total of 103 white stars which were due to interactions of primary tracks with free protons and with quasi-free-nucleons in the emulsions. In order to separate the interactions between the primary protons and the free protons, from the interactions of primary protons with the protons bound to nuclei, we followed the following criteria:

i) an even number of secondary prongs;

ii) absence of a recoil blob or of a low energy electron (*i.e.*, a clean event);

iii) the angles of emission and the energies of the secondary particles should not contradict the kinetics of proton collision between a proton and a proton at rest.

After making correction for the possible contamination we thus classified 63 events a spossible proton-free-proton collisions, and 30 events as proton quasi-free-neutron interaction. Assuming that the number of collisions of protons with quasi-free-protons in a nucleus is equal to the number of collisions with quasi-free neutrons, we get thus 33 events due to inelastic proton-free-proton, [later called (p-f-p) collisions]. From this analysis we are unable to decide in detail which 33 events out of 63 possible (p-f-p) inelastic collisions are true (p-f-p) collisions. In fact, in general, one cannot identify individually the inelastic interactions with hydrogen nuclei in nuclear emulsion. The average multiplicity for charged secondary particles of the total 63 possible (p-f-p) interactions was calculated to be n_s (inelastic) 4.0 ± 0.5.

In order to identify the events which were due to elastic (p-f-p) collisions we made the usual tests on two-prong white stars, for coplanarity and for scattering angles of the shower particles which must be consistent with the kinematics of elastic scattering, *i.e.*, $\text{tg } \psi_1 \cdot \text{tg } \psi_2 = 1 - (v/c)^2$ where v is the velocity of the CMS with respect to the laboratory system. We found thus 5 possible elastic (p-f-p) scattering events. The cross-section value for inelastic (p-f-p) interaction after making correction for possible contamination is about 36 mb and for elastic (p-f-p) scattering is about 4.9 mb.

In order to transform the laboratory angles (ψ_L) of the shower particles of clean events with $N_h = 0$ and 1 and having an even number of secondary prongs into the CMS system, we used the following simple transformation:

$$\text{tg } \psi_L = [1 - (v/v)^2]^{\frac{1}{2}} \text{ tg } (\psi_c/2) .$$

In Fig. 3 is shown the individual distributions of 2, 4 and 6 prong events.

Fig. 3. – The angular distribution in the center-of-mass system of events with $N_h = 0$ and 1; for the individual distribution of 2, 4 and 6 prong stars. ·········· 2 Prong stars; ——— 4 Prong stars; – – – – 6 Prong stars.

Practically all these distributions show anisotropy.

We may mention here that the average multiplicity is slightly less than one would expect at this energy and it is due to a slight contamination of secondaries present in the primary beam.

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We thank Prof. M. F. KAPLON for his criticism and also for his kindness in supplying us with the emulsion plates.

Possible Bound Σ-Λ System (*).

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Recent experimental evidences tend to favor an assignment of odd relative $\Sigma\Lambda$ parity [1,2]. The Σ can then be considered as a bound system of Λ and π in an S state, with the scalar $\Sigma\Lambda\pi$ coupling constant being of the order of 1 [1,3]. It is indeed an interesting feature of the bound state picture that the coupling constant is more or less determined by the binding energy if the latter is small enough.

Some interesting predictions can be made for processes involving Σ and Λ once we take the above picture in a serious way [3-5]. We would like to report here another such consequence which naturally follows from the same assumptions. This is the interaction between a Σ and a Λ through the

---

exchange of a pion. (Coupling to the ΞN system is neglected for the moment). Because of the scalar coupling, one gets in the static limit a Yukawa potential with an exchange character

$$(1) \qquad V = g^2 \frac{\exp\left[-\mu r\right]}{r} P_x P_s \,.$$

where μ is the meson mass, and P_x and P_s are the space and spin exchange operators. Although g^2 is ≈ 1, the higher order potentials (many-meson exchanges) should not be important unlike the nucleon-nucleon case.

We see from eq. (1) that V is negative for $P_x P_s = -1$, ie. for the states ^1S, ^3P, etc., and consequently there arises a possiblility of bound $\Sigma\Lambda$ formation. In terms of the bound state picture of Σ, one may describe the $\Sigma\Lambda$ system as a lambda molecule $\Lambda_\pi\Lambda$ bound by virtue of a pion. This will be possible if the Pauli principle does not repel the two Λ's, namely if $P_x P_s = -1$.

The binding energy predicted from eq. (1) turns out to be surprisingly large. In order to get a general picture, we will replace the Yukawa by the Hulthén potential. The binding energy for the S state is then given

[1] Y. Nambu and J. J. Sakurai: *Phys. Rev. Lett.*, **6**, 377 (1961). See also S. Barshay: *Phys. Rev. Lett.*, **1**, 97 (1958).

[2] J. J. Sakurai: *Nuovo Cimento*, **20**, 1212 (1961).

[3] J. Bernstein and R. Oehme: *Phys. Rev. Lett.*, **6**, 639 (1961).

[4] J. Franklin, R. C. King and S. F. Tuan: to be published.

[5] S. Barshay and H. Pendleton: II. to be published.

by ([6])

$$(2) \quad \begin{cases} \varepsilon_n = \dfrac{\mu^2}{8m}\left(\dfrac{b}{n} - n\right)^2 , \quad n = 1, 2, ..., \\[2mm] b = 2g^2 m/\mu , \end{cases}$$

where m is the reduced mass (≈ 600 MeV), and n is the principal quantum number. The result is listed in Table I for $g^2 = 0.5$, 1 and 1.5.

TABLE I. – *The binding energies ε_n (MeV) according to the Hulthén potential.*

	$g^2 = 0.5$	$g^2 = 1$	$g^2 = 1.5$
ε_1	40	220	510
ε_2	0.04	17	70
ε_3	—	—	4

This estimation, however, is not physically reliable when the binding energy is as big as a few hundred MeV. For the size of the system $\approx 1/\sqrt{2m\varepsilon}$ would then be much smaller than the range of the potential, which means that the result would depend critically on the details of the unknown short range part of the interaction. For example, if there is an analog of the hard core of the nucleon-nucleon case, the binding energy should be drastically reduced. We have estimated such an effect of the hard core by using a simple trial wave function

$$\psi(r) = (1 - r/r_0) \exp[-\lambda r] .$$

where r_0 is the core radius. The resulting lower bound to the ground state binding energy is plotted in Fig. 1 for various values of r_0 and g^2. One finds, for example, that with $g^2 = 1$ and

$$r_0 = \tfrac{3}{2} m = 0.5 \cdot 10^{-13} \text{ cm} ,$$

([6]) L. ROSENFELD: *Nuclear Forces*, Vol. 1 (Amsterdam, 1948), p. 76.

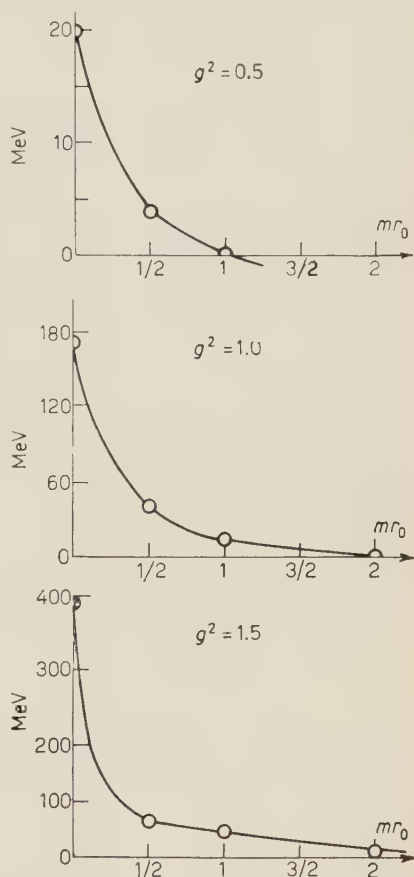the binding energy is of the order of several MeV.

Fig. 1. – Binding energy *vs.* core radius for different coupling constants (variational estimation).

Thus the binding energy is very sensitive to g^2 and the details of the short range potential. But it seems quite likely that there is at any rate a strong tendency for a bound state formation. We will now consider its implications.

We first take note of the fact that the $\Sigma\Lambda$ system has strangeness $S = -2$ and isospin $T = 1$, and is unstable by 50 MeV relative to ΞN, neglecting the binding effects. $\Sigma^0\Lambda$ is also unstable

by 70 MeV relative to $\Lambda\Lambda$ via electromagnetic processes. The coupling to the $\Xi\mathcal{N}$ channel, which was neglected in the above considerations, occurs through a K meson exchange, which may or may not be too important, and will depend of course on the coupling constants and the Ξ parity. Theoretically, our model does not predict anything about the $\Xi\mathcal{N}$ system [7]. If the binding energy of $\Sigma\Lambda$ is small, the bound state will actually be observed as a resonance in the $\Xi\mathcal{N}$ channel. When the binding becomes large enough ($\varepsilon > 50$ MeV), it will be able to decay into $\Sigma\mathcal{N}$ or $\Lambda\mathcal{N}$ via weak interactions. (We know empirically that $\Sigma\mathcal{N}$ and $\Lambda\mathcal{N}$ interactions are weak.) Even these $\Delta S = 1$ processes become energetically forbidden for $\Sigma^-\Lambda$ (and for $\Sigma^+\Lambda$ under $\Delta S/\Delta Q = 1$) if $\varepsilon > 180$ MeV and for $\Sigma^+\Lambda$ if $\varepsilon > 250$ MeV. In such cases, the stability of $\Sigma\Lambda$ would depend very much on the presence or absence of $\Delta S = 2$ weak interactions.

Experimentally, there are some evidences for the formation of $S = -2$

hypernuclei [8,9], but little is known about the two-particle system we are concerned with. A possible way to test our conclusions would be to study the processes

$$K^- + d \to K^+ + \Xi^- + n \,,$$
$$K^+ + \Sigma + \Lambda \,.$$
$$K^+ + (\Sigma\Lambda) \,, \text{ etc. } \,.$$
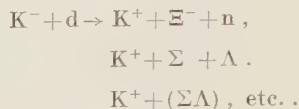
For example, the Q-value distribution fo the Ξ^-n pair may show a resonance structure due to $(\Sigma\Lambda)$. If the $\Sigma\Lambda$ forces are not strong enough to cause binding there may still be a low energy resonance above the $\Sigma\Lambda$ threshold.

Should the $\Sigma\Lambda$ binding be large enough to be stable against $\Xi\mathcal{N}$ conversion, the inverse reaction, namely the radiative capture of Ξ by $\mathcal{N}$,

$$\Xi^- + \mathcal{N} \to (\Sigma\Lambda) + \gamma \,,$$
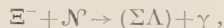
could also occur in competition with the processes

$$\Xi + \mathcal{N} \to \Lambda + \Lambda \,.$$
$$\to \Lambda + \Lambda + \gamma \,.$$
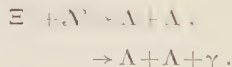
[7] Some theoretical considerations on the $\Xi\mathcal{N}$ system have been made by J. Szimanski: *Nuovo Cimento*, **11**, 730 (1959). See also L. B. Okun', I. Ya. Pomeranchuk and I. M. Shmushkevitch: *Žurn. Éksp. Teor. Fiz.*, **34**, 1246 (1958); S. B. Treiman: *Phys. Rev.*, **113**, 355 (1959).

[8] W. H. Barkas, N. N. Biswas, D. A. Delise, J. N. Dyer, H. H. Heckman and F. M. Smith: *Phys. Rev. Lett.*, **2**, 466 (1959).

[9] D. H. Wilkinson, S. J. St. Lorant, D. K. Robinson and S. Lokanathan: *Phys. Rev. Lett.*, **3**, 347 (1959).

Asymptotic Condition in the Lorentz Gauge (*).

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It has been pointed out by EVANS and FULTON [1] that the asymptotic condition, in its present formulation [2,3], is incompatible with quantum electrodynamics in the Lorentz gauge [3]. The asymptotic vector potential, say A_{in}^μ, is postulated to be a free field, so that

$$(1) \qquad\qquad \Box A_{in}^\mu = 0 \,,$$

$$(2) \qquad\qquad [A_{in}^\mu(x), A_{in}^\nu(y)] = ig^{\mu\nu} D(x-y) \,.$$

On the other hand, if the interacting Heisenberg fields $A^\mu(x)$ are computed on the basis of the electrodynamic Lagrangian, and if one defines the usual weak limit

$$(3) \qquad \mathscr{A}_{in}^\mu(x) = \lim_{y^0 \to -\infty} \int \mathrm{d}^3 y \left[-A^\mu(y) \frac{\partial}{\partial y^0} D(x-y) + D(x-y) \frac{\partial}{\partial y^0} A^\mu(y) \right] \,,$$

then one obtains

$$(4) \qquad\qquad \Box \mathscr{A}_{in}^\mu(x) = 0 \,,$$

$$(5) \qquad\qquad [\mathscr{A}_{in}^\mu(x), \mathscr{A}_{in}^\nu(y)] = i(g^{\mu\nu} - M \partial^\mu \partial^\nu) D(x-y) \,.$$

The number M does not vanish in perturbation theory, and also in a more general treatment [3] has been shown to be non-zero. Hence

$$(6) \qquad\qquad A_{in}^\mu \neq \mathscr{A}_{in}^\mu \,.$$

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[1] L. E. EVANS and T. FULTON: preprint.
[2] H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **1**, 205 (1955).
[3] G. KÄLLÉN: *Handb. d. Phys.*, vol. **5**, pt. 1 (Berlin, 1958).

The purpose of this note is to observe that if one reformulates the asymptotic condition to read

$$(7) \qquad A_{\text{in}}^{\mu}(x) = \mathscr{A}_{\text{in}}^{\mu}(x) + \tfrac{1}{2} M \, \partial^{\mu} \partial_{\nu} \mathscr{A}_{\text{in}}^{\nu}(x) \ ,$$

with $\mathscr{A}_{\text{in}}^{\mu}$ obtained from (3), then the desired relations (1), (2) follow immediately from (5). It therefore seems that (3), (7) is the correct limit in the case of the interacting photon.

The reduction formula for the S matrix is unaffected by this gauge transformation. Indeed, if $f^{\mu}(x)$ is a physical free photon wave function, satisfying the Lorentz condition $\partial \cdot f = 0$, then the creation operator for incoming physical states

$$(8) \qquad a_{\text{in}}^{f*} = i \int \mathrm{d}^3x \left[- \mathscr{A}_{\text{in}}^{\mu}(x) \frac{\partial}{\partial x^0} f_{\mu}(x) + f^{\mu}(x) \frac{\partial}{\partial x^0} \mathscr{A}_{\text{in}\,\mu}(x) \right]$$

and its Hermitian adjoint a_{in}^{f} are unchanged under the substitution $\mathscr{A} \to A$.

By considering a complete orthogonal set of functions f^{μ} which do not necessarily satisfy the Lorentz condition one may now use (8) to construct orthogonal (and presumably complete) sets of asymptotic states, including the unphysical ones.

Existence of a Stable Neutral Meson.

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In this note we should like to describe a new particle, a neutral meson, which seems to be stable. We shall first describe its strong and weak interactions, then indicate experiments by means of which this particle might be observed, (including some evidence which suggested its existence), and finally suggest a scheme under which such a particle might arise. Our major point is that if a scalar or pseudoscalar boson is odd under charge conjugation, it is stable or extremely long lived in the absence of neutral lepton currents.

Let us first consider the possible strong interactions of a particle of this type which we shall call for the present X^0. Since the strong interactions are invariant under charge conjugation, the only Yukawa coupling is a derivative coupling of the form $\overline{\psi}\gamma_\mu\psi(\partial X^0/\partial x_\mu)$, (which is possible only if the particle is a scalar). This can be transformed away, so that any strong interactions must be of a « pair » type *e.g.*

$$X^0 X^0 \overline{\psi}\psi, \quad iX^0(\overline{A}\overline{K}N - \overline{N}KA),$$

or

$$iX^0(\overline{\boldsymbol{\Sigma}}\cdot\overline{K}\boldsymbol{\tau}N - \overline{N}\boldsymbol{\tau}K\cdot\boldsymbol{\Sigma})\,(^2).$$

(²) In the reactions

$$K^- + p \rightarrow \begin{pmatrix}\Sigma^0 \\ A^0\end{pmatrix} + \pi^0, \quad \overline{K}^0 + n \rightarrow \begin{pmatrix}\Sigma^0 \\ A^0\end{pmatrix} + \pi^0,$$

$$K^- + n \rightarrow \Sigma^- + \pi^0,$$

there would be an apparent violation of charge independence if the latter coupling were present since the X^0 and π^0 would not be distinguished. Further, a similar situation would hold for the production of an additional π^0 in associated production. The experimental data indicate that the rate for production of X^0 is less than 5 % that for the π^0.

Therefore, if produced via the strong interactions, X^0 is always produced in pairs, or in conjunction with other particles and as we shall see, it is stable or very long lived, it is not likely to have been detected among the strong interactions.

Since the weak interactions do not conserve C, almost any coupling would be possible. One would not, however, expect X^0 to be coupled to the leptons since no neutral lepton currents have been observed. X^0 could be coupled via an intermediate boson W^0 or directly with other heavy particles. If X^0 does not interact strongly it would be a sort of leptonic boson.

Let us turn to a consideration of possible decay modes. If the mass of X^0 is smaller than or equal to the π mass the absence of a neutral leptonic current prevents decays via the weak interactions. The only remaining mode is the photonic decay. Since X^0 is odd under charge conjugation it can only decay into an odd number of photons. The only requirements on the matrix elements are produced by gauge invariance and Bose statistics. The matrix elements must contain the electromagnetic field tensor an odd number of times, the field X^0, and all possible derivatives. An elementary calculation indicates that all the local, effective matrix elements vanish for three-photon decay and, thus, the photon decay is strongly suppressed [3]. Actually, the restriction on the possible couplings provides a more simple method for demonstrating the strong suppression of photonic decays. Since electromagnetic

decays of a neutral particle proceed only through the strong interactions, and the Yukawa coupling can be transformed away, the « pair » coupling only is operative. The first of these leads to no decay and the second to a strongly momentum-dependent matrix element [4]. Assuming only Yukawa couplings for the strong interactions, the X^0 is stable.

Now we are faced with the problem of indicating a means for detecting the X^0. The decay of the K^+-particles provide the best arena for the detection of the X^0 (we must point out that we were led to consider the X^0 because of an anomaly in the K^+ experiments). $K_{\pi2}$ decay is forbidden by the $\Delta I = \frac{1}{2}$ whereas the decay $K^+ \to \pi^+ + X^0$ is allowed. If the mass of X^0 is close to the mass of π, reaction $K^+ \to \pi^+ + X^0$ would not be kinematically distinguishable from $K_{\pi2}$. The difference would arise only if one observed the decay of π^0. An experiment of this type has been carried out in the xenon bubble chamber and indicates a serious discrepancy with the emulsion results. The $K_{\pi2}$ branching ratio is $\frac{2}{3}$ that of the emulsion experiments [5]. If both experiments are accurate the discrepancy can easily be accounted for by a particle with the properties of X^0 and would indicate that the mass must be very close to that of a π^0. The X^0 might also be produced in hyperon decay through

$$\Lambda^0 \to n + X^0, \qquad \Sigma^+ \to p + X^0 .$$

Finally, we should like to consider a scheme by which a particle with the

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[3] These have the following form:

$$X^0 F_{\mu\nu} F_{v\sigma} F_{\sigma\mu} , \quad \frac{\partial X^0}{\partial X_\mu} F_{\mu\nu} \frac{\partial F_{\lambda\sigma}}{\partial x_\nu} F_{\lambda\sigma} , \text{ etc.}$$

By manipulating the indices and making use of the anti-symmetry of $F_{\mu\nu}$; they can be shown to vanish.

[4] The coupling argument holds both for scalar and pseudoscalar particles. In fact, the relative stability of the particle is independent of the parity.

[5] G. ALEXANDER, R. H. W. JOHNSTON and C. O'CEALLAIGH: Nuovo Cimento, 6, 478 (1957); H. S. BRIDGE, D. H. PERKINS, J. R. PETERSON, D. H. STORK and M. N. WHITEHEAD: Nuovo Cimento, 4, 834 (1956); B. ROE and D. SINCLAIR: private communication.

properties of the X^0 might arise. Suppose the π's behave like a doublet for the weak interactions. The π's would have the form:

$$\begin{bmatrix} \pi^+ \\ \varphi^0 \end{bmatrix}, \quad \begin{bmatrix} \overline{\varphi}^0 \\ \pi^- \end{bmatrix}.$$

We assume that under C, $C\varphi^0 C^{-1} = \overline{\varphi}^0$ and under P, $P\varphi^0 P^{-1} = -\varphi^0$. However, the π's also interact strongly and since the strong interactions are CP invariant, only those combinations of φ^0 and $\overline{\varphi}^0$ which are eigenstates of CP are permitted [6].

These are

$$\frac{\varphi^0 + \overline{\varphi}^0}{\sqrt{2}} \quad \text{and} \quad \frac{\varphi^0 - \overline{\varphi}^0}{\sqrt{2}}.$$

The first of these is the π^0 which forms a triplet with π^+, π^-. By an appropriate choice of coupling constants, the interaction can be made charge independent. The latter is the singlet X^0 and would, thus, have odd parity. However, the X^0 fits into other schemes of the strong interactions as well, such as the 4-dimensional one of SHIMAMOTO [7]. We are at present continuing the study of the properties of this particle and an experiment is being contemplated by ROE and SINCLAIR which would attempt to detect both the π^+ and π^0 in the K^+ decay.

* * *

[6] Cf. G. FEINBERG: *Phys. Rev.*, **108**, 878 (1957).

[7] Y. SHIMAMOTO: private communication.

Consequences of Unitary Symmetry
for Weak and Electromagnetic Transitions.

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Recent papers [1] have dealt with the introduction of unitary symmetry, *i.e.* invariance under the three-dimensional unitary group, as a convenient approximation in the theory of strong interaction. The strong-interaction Lagrangian is assumed to consist of a part invariant under the unitary group, plus a « correction » breaking the invariance.

In this note we shall examine the properties that follow, for weak and electromagnetic amplitudes, from this hypothesis, together with the violent assumption that the symmetry-breaking « corrections » can be neglected. We do not know under what conditions this last hypothesis can be applied. It might be applicable in the high-energy region — or, better, we know for sure that it is not generally applicable at low energies.

The consequences of violent assumptions are usually far-reaching. Thus we find that the K^0 (or $\bar{K}^0$) electromagnetic form factors must be zero, the charged K form factors must be equal to the charged pion form-factors; there are simple stringent relations betwen the form factors of the baryons (already given by Coleman and Glashow [2]), of the vector mesons, between the electromagnetic amplitudes from vector to pseudoscalar mesons, between the amplitudes of χ^0 decay and π^0 decay, and between weak interaction amplitudes. These relations depend in part on the particular representation adopted for the baryons.

For instance according to the « eightfold way » the Λ form-factors are one-half of the neutron form factors. They must instead be equal in the Sakata represen-

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(1) S. Ogawa: *Progr. Theor. Phys.*, **21**, 209 (1959); Y. Yamagouchi: *Progr. Theor. Phys. Suppl.*, **11**, 37 (1959); M. Ikada, S. Ogawa and Y. Ohnuki: *Progr. Theor. Phys.*, **22**, 715 (1959); **23**, 1073 (1960); I. E. Wess: *Nuovo Cimento*, **15**, 52 (1960); Z. Meki, M. Nakagawa, Y. Ohnuki and S. Sakata: *Progr. Theor. Phys.*, **23**, 1174 (1960); M. Ikeda, Y. Miyachi and S. Ogawa: *Progr. Theor. Phys.*, **24**, 569 (1960); M. Gell-Mann: to be published in *Phys. Rev.*; A. Salam and J. C. Ward: *Nuovo Cimento*, **20**, 419 (1961); Y. Neeman: preprint.

(2) S. Coleman and S. L. Glashow: *Phys. Rev. Lett.*, **6**, 1423 (1961).

tation. If this is true also at low energy a measurement of the magnetic moment would distinguish among the two cases.

1. – The unitary group in three dimensions has 8 generators F_m. Their commutation rules are given, in terms of the totally antisymmetric tensor $(^3)$ f_{mnl}, by

$$(1) \qquad\qquad [F_m, F_n] = i f_{mnl} F_l \ .$$

In correspondence to each F_m there is a current $j_m(x)$, conserved as long as unitary simmetry holds. The currents transform according to

$$(2) \qquad\qquad [F_m, j_n(x)] = i f_{mnl} j_l(x) \ .$$

In the theory of Gell-Mann and Neeman (eightfold way), where baryons transform according to the 8-dimensional rapresentation, the electromagnetic current, $j(x)$, is given by

$$(3) \qquad\qquad j(x) = j_3(x) + \frac{1}{\sqrt{3}} j_8(x) \ .$$

From (2) and (3) one sees that F_3, F_8, F_6, and F_7 are the generators of the subgroup that leaves $j(x)$ invariant. Conservation of F_3 and F_8 expresses conservation of I_3 and of hypercharge. We thus limit our considerations to F_6 (F_7 gives the same results). From the identity

$$\langle A \,|\, [F_6, \, j(x_1) \cdots j(x_n)] \,|\, B \rangle = 0 \ ;$$

where $j(x_1) \cdots j(x_n)$ is any product of n currents, we derive, if A and B are one-particle states

$$(4) \qquad\qquad \langle 0 \,|\, [\psi_A, F_6] j \cdots j \,|\, B \rangle = \langle A \,|\, j \cdots j [F_6, \psi_B^+] \,|\, 0 \rangle \ ,$$

where ψ^+, ψ are the relevant creation or annihilation operators, and we have briefly denoted by $j \cdots j$ the product of the currents. In Gell-Mann's « eightfold way » both baryons and mesons are assumed to transform according to the 8-dimensional representation. Therefore, for a suitable choice of the ψ's,

$$(5) \qquad\qquad [\psi_m, F_n] = i f_{mnl} \psi_l \ ,$$

and thus (4) relates directly different matrix elements between one particle states. If we apply it to the mesons we obtain, for instance, the following conclusions:
— the K^0 (or $\bar{K}^0$) form factor is always zero (we use also charge conjugation invariance);
— the form factor of K^+ (K^-) is equal to that of π^+ (π^-);
— the amplitude for Compton scattering on $K^{\pm}$ is equal to the same amplitude of a $\pi^{\pm}$. For the corresponding amplitudes on neutral mesons one has relations

of the kind:

$$\sqrt{3}\langle K^0 | jj | K^0 \rangle = \langle \chi^0 | jj | \pi^0 \rangle - \sqrt{3}\langle \chi^0 | jj | \chi^0 \rangle \; ;$$

$$-\langle K^0 | jj | K^0 \rangle = \langle \pi^0 | jj | \pi^0 \rangle - \sqrt{3}\langle \pi^0 | jj | \chi^0 \rangle \; .$$

— The amplitude for $\chi^0 \to \gamma + \gamma$ is $1/\sqrt{3}$ times the amplitude for $\pi^0 \to \gamma + \gamma$ etc. By applying (4) to the baryons one finds

$$(6) \quad \begin{cases} \langle \Sigma^+ | j \cdots j | \Sigma^+ \rangle = \langle p | j \cdots j | p \rangle \; , \\[1mm] \langle \Sigma^- | j \cdots j | \Sigma^- \rangle = \langle \Xi^- | j \cdots j | \Xi^- \rangle \; , \\[1mm] \langle \Xi^0 | j \cdots j | \Xi^0 \rangle = \langle n | j \cdots j | n \rangle - \\[1mm] -\dfrac{1}{\sqrt{3}} \langle \Sigma^0 | j \cdots j | \Lambda \rangle = \langle n | j \cdots j | n \rangle - \langle \Lambda | j \cdots j | \Lambda \rangle \\[1mm] -\sqrt{3}\langle \Lambda | j \cdots j | \Sigma^0 \rangle = \langle n | j \cdots j | n \rangle - \langle \Sigma^0 | j \cdots j | \Sigma^0 \rangle \; , \end{cases}$$

A relation between the electromagnetic mass splittings,

$$\delta m_{\Xi^-} - \delta m_{\Xi^0} = \delta m_p - \delta m_n + \delta m_{\Sigma^-} - \delta m_{\Sigma^+} \, ,$$

given by COLEMAN and GLASHOW [2], is contained in (6).

If the current product $j \cdots j$ reduces to a single j one can make further use of the transformation properties of j. A matrix element $\langle A | j_m | B \rangle$ where A, B, j_m all transform according to the eight-dimensional representation can be decomposed as

$$(7) \qquad \langle A | jm | B \rangle = i f_{ABm}\, \mathcal{O} + d_{ABm}\, \mathcal{E} \; ,$$

in terms of the totally antisymmetric tensor f, of the totally symmetric tensor d[3], and of the quantities $\mathcal{O}$ and $\mathcal{E}$.

The identity (7) is similar to the familiar Wigner-Eckart theorem for space rotations. The quantities $\mathcal{O}$ and $\mathcal{E}$ play the role of the so-called reduced matrix elements, and f and d of Clebsch-Gordon coefficients.

The reason why one has two reduced matrix elements in this case is that the reduction of the direct product $8 \times 8 \times 8$ contains the representation 1 twice.

Applying (7) to (3) one has

$$(8) \qquad \langle A | j | B \rangle = i\left(f_{AB3} + \frac{1}{\sqrt{3}} f_{AB8}\right) \mathcal{O} + \left(d_{AB3} + \frac{1}{\sqrt{3}} d_{AB8}\right) \mathcal{E} \; .$$

It is instructive to compare with the corresponding situation when only charge independence is assumed. In that case one has two independent matrix elements,

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[3] A table of the elements of f_{mnl} as well as of d_{mnl} is given in the paper by GELL-MANN [see [1]].

usually called the scalar part and the vector part, which originate directly from the decomposition of j, analogous to (3), into an isovector and an isoscalar part. Here instead, j_3 and j_8 transform both in the same way, according to (2), but each of them originates two reduced matrix elements.

With (8) one fonds directly

$$(9)\quad\begin{cases}
(6) & \langle \Sigma^0 | j | \Sigma^0 \rangle = \tfrac{1}{3}\mathscr{E}\,, \\[6pt]
(3) & \langle \varLambda^0 | j | \varLambda^0 \rangle = -\tfrac{1}{3}\mathscr{E}\,, \\[6pt]
(4) & \langle \Sigma^0 | j | \varLambda^0 \rangle = \dfrac{1}{\sqrt{3}}\mathscr{E}\,, \\[6pt]
(7) & \langle \Sigma^- | j | \Sigma^- \rangle = \tfrac{1}{3}\mathscr{E} - \mathscr{O}\,, \\[6pt]
(9) & \langle \Xi^- | j\ \Xi^- \rangle = \tfrac{1}{3}\mathscr{E} - \mathscr{O}\,, \\[6pt]
(8) & \langle \Xi^0 | j | \Xi^0 \rangle = -\tfrac{2}{3}\mathscr{E}\,, \\[6pt]
(1) & \langle p | j | p \rangle\ \ = \tfrac{1}{3}\mathscr{E} + \mathscr{O}\,, \\[6pt]
(2) & \langle n | j | n \rangle\ \ = -\tfrac{2}{3}\mathscr{E}\,, \\[6pt]
(5) & \langle \Sigma^+ | j | \Sigma^+ \rangle = \tfrac{1}{3}\mathscr{E} + \mathscr{O}\,.
\end{cases}$$

From (9) one obtains the relations between the anomalous magnetic moments given by COLEMAN and GLASHOW [2,4]. We recall that one has (denoting explicitly the tensor indices)

$$\mathscr{O}^\mu = \bar{u}(p_f)\,\mathscr{O}_1(K^2)\gamma^\mu + \mathscr{O}_2(K^2)\sigma^{\mu\nu}K^\nu]\,u(p_i)\,,$$

with $K = p_f - p_i$, and similarly for $\mathscr{E}^\mu$.

Relations similar to (9) hold for the form factors of the postulated vector mesons or for the amplitudes of radiative transitions between vector mesons and pseudo-scalar mesons. Thus one finds, for instance,

$$\langle \pi_0' | j | \chi_0 \rangle = \langle \chi_0' | j | \pi^0 \rangle = -\frac{1}{\sqrt{3}}\langle \chi_0' | j | \chi_0 \rangle = \frac{1}{3}\langle \chi_0' | j | \pi_0 \rangle = -\frac{2}{\sqrt{3}}\langle K_0' | j | K^0 \rangle\,,$$

$$\langle \pi'^+ | j | \pi^+ \rangle = \langle K'^+ | j | K^+ \rangle\,,$$

where, for instance, π_0' is the vector meson with the same isospin properties of π^0. Furthermore K'^+ (K'^-) has the same form factors as π'^+ (π'^-), and the form factors of neutral vector mesons are all zero.

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[4] Consistency with the relations of Coleman and Glashow requires an additional minus-sign in ther definition of the Σ-$\varLambda$ transition moment.

2. – If one assumes that the weak currents are simply linear combinations of the currents j_m ([5]), one can apply (7) to derive relations between amplitudes for leptonic decays, always in the same spirit of neglecting that part of the strong Lagrangian that violates unitary simmetry. Thus, the $\Delta S = +1$, $\Delta Q = +1$ weak current could be of the form $g(j_4 + ij_5)$ where g is a constant. One then has the decomposition

$$(10) \qquad g\langle A \,|\, j_4 + ij_5 \,|\, B\rangle = (if_{AB4} - f_{AB5})\,\mathcal{O}' + (d_{AB4} + id_{AB5})\,\mathscr{E}',$$

where $\mathcal{O}'$ and $\mathscr{E}'$ are $g\mathcal{O}$ and $g\mathscr{E}$.

If one assumes universality in the coupling of the weak currents to the leptons the $\Delta S = 0$, $\Delta Q = +1$ weak current would be $g(j_1 + ij_2)$, with the same g as in (10). But then the rates for hyperon leptonic decays would be much larger than observed. Therefore the use of the universality hypothesis is, at least, inconvenient, in such a scheme; of course, the hypothesis may be true, but masked by strong renormalization effects. We do not therefore insist on the relations between matrix elements of different currents. From (10) one finds

$$(11) \quad \left|
\begin{aligned}
&g\langle \Xi^- \,|\, j_4 + ij_5 \,|\, \Lambda\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{3}\,\mathcal{O}' - \frac{1}{\sqrt{3}}\,\mathscr{E}'\right), \\[2mm]
&g\langle \Sigma^- \, j_4 + ij_5 \,|\, n\rangle \;= -\mathcal{O}' + \mathscr{E}', \\[2mm]
&g\langle \Sigma^0 \,|\, j_4 + ij_5 \,|\, p\rangle \;= \frac{1}{\sqrt{2}}\,(-\mathcal{O}' + \mathscr{E}'), \\[2mm]
&g\langle \Lambda \,|\, j_4 + ij_5 \,|\, p\rangle \;= \frac{1}{\sqrt{2}}\left(-\sqrt{3}\,\mathcal{O}' - \frac{1}{\sqrt{3}}\,\mathscr{E}'\right), \\[2mm]
&g\langle \Xi^- \,|\, j_4 + ij_5 \,|\, \Sigma^0\rangle \;= \frac{1}{\sqrt{2}}\,(\mathcal{O}' + \mathscr{E}'), \\[2mm]
&g\langle \Xi^0 \,|\, j_4 + ij_5 \,|\, \Sigma^+\rangle = \mathcal{O}' + \mathscr{E}'.
\end{aligned}
\right.$$

3. Still in the same spirit of neglecting violations of unitary simmetry we can easily establish that, in the limit of zero momentum transfer, $\mathscr{E}' \to 0$ (i.e. the form factor multiplying γ_μ in the expansion of $\mathscr{E}'$ is zero for $K^2 = 0$). In fact in the limit of zero momentum transfer the relevant matrix element is proportional to $\langle A \,|\, F_4 + iF_5 \,|\, B\rangle$, since the generators F_m are also the space integrals of the fourth component of j_n (and, of course, are conserved if j_m is divergenceless). However

$$(12) \qquad \langle A \,|\, F_4 + iF_5 \,|\, B\rangle = \langle 0 \,|\, [\psi_A, F_4 + iF_5] \,|\, B\rangle = if_{AB4} - f_{AB5},$$

showing that $\mathscr{E}'(0) = 0$ and $\mathcal{O}'(0) = g$.

([5]) If strange currents violating $\Delta S = \Delta Q$ exist (preliminary report from the Padua-Wisconsin group) this possibility is lost, at least in the framework of the three dimensional unitary group.

4. – There are other versions of the models based on unitary symmetry differing mainly in the representation of the baryons. In the model by Gell-Mann and Neeman the eight known baryons are the basis of an eight-dimensional representation. In the original Sakata model (¹) three baryons p, n, Λ, are the basis of a three dimensional representation, while other baryons belong to higher representations (to the 15-dimensional, or to both the 15 — and the 6 — dimensional representation). One can easily extend the considerations we have made in the previous sections to the Sakata model. One has however to be careful in identifying properly the currents. The electromagnetic current in the Sakata model is no longer given by (3) but it is given instead by

$$(13) \qquad j(x) = j_3(x) + \frac{1}{\sqrt{3}} j_8(x) + \frac{1}{3} j_0(x) \; .$$

Note the addition of $\frac{1}{3} j_0(x)$, proportional to the baryonic current, which belongs to the one-dimensional representation, and therefore does not transform according to (2). This circumstance again brings two independent « reduced » matrix elements and we find the relation

$$(14) \qquad \langle n | j | n \rangle = \langle \Lambda | j | \Lambda \rangle \; ,$$

i.e., neutron and Λ have the same form factors and anomalous moment.

In contrast to this results, the eightfold way gives $\langle n | j | n \rangle = 2 \langle \Lambda | j | \Lambda \rangle$. The Λ anomalous magnetic moment will presumably be measured rather soon by precession in a strong magnetic field (⁶).

(*) This experiment is under development at Brookhaven and at CERN.

LIBRI RICEVUTI E RECENSIONI

R. H. Dicke and J. P. Wittke —
Introduction to Quantum Mechanics; Addison-Wesley Publishing
Company Inc., Reading, Mass.,
USA; pp. XI-370, prezzo $ 8.75.

Il contenuto del volume corrisponde
più o meno ai programmi dei nostri corsi
universitari di meccanica quantistica.
Nei primi tre capitoli gli autori svolgono
il necessario esame critico dell'applicabilità di concetti classici ai fenomeni
microscopici. Inoltre viene mostrato,
senza ancora pretendere ad una formulazione rigorosa, come l'uso di concetti
tipicamente ondulatori permette di interpretare fenomeni inspiegabili dal punto
di vista classico. D'altra parte un formalismo sufficiente deve anche essere
capace di esibire aspetti corpuscolari, da
cui la necessità di incorporare nella
descrizione un principio di complementarità. I successivi sette capitoli contengono una esposizione dettagliata e consistente della meccanica ondulatoria di
Schrödinger. Infine gli ultimi otto capitoli contengono la trattazione della meccanica quantistica nello spazio di Hilbert,
lo studio dei metodi di approssimazione
e le loro applicazioni, la discussione del
problema di particelle identiche ed una
molto bella introduzione alla meccanica
statistica quantica interamente basata
sulla nozione di matrice di densità.

Vorremmo brevemente commentare
su due punti. Anzitutto è chiaro dalla
suddivisione dell'opera che gli autori si
sono sforzati di sviluppare una prima
parte della trattazione su un tono più
semplice ed accessibile. Si tratta della
meccanica ondulatoria alla Schrödinger,
trattata nei primi dodici capitoli. Il punto
di vista è il seguente. L'uso di concetti
di meccanica ondulatoria in fisica comincia subito a divenire necessario allo
studente, dopo avere studiato la fisica
macroscopica, per qualunque ulteriore
studio. È necessario che se ne offra una
esposizione semplice e tuttavia sufficiente,
quanto prima nel corso dell'insegnamento
universitario. La discussione formale della
meccanica quantistica può essere presentata in un tempo successivo. Vi saranno
certamente obbiezioni a questo punto di
vista ma riteniamo che la sua adozione
stia diventando inevitabile. La meccanica ondulatoria alla Schrödinger va
insegnata quanto prima possibile nel
curriculum universitario, con la riserva
di una successiva discussione più approfondita. Il secondo punto che vorremmo commentare riguarda l'insegnamento della meccanica quantistica relativistica. Gli autori di questo volume in
recensione non includono alcuna discussione di meccanica quantistica relativistica allo scopo di evitare le « complicazioni e gli aspetti insoddisfacenti della
teoria dei campi ». Esisteva in passato,
in parte per giustificabili motivi storici
ed applicativi, la deprecabile, a nostro
avviso, abitudine di includere alla fine
del corso di meccanica quantistica la
discussione dell'equazione di Dirac. Ora
è vero che l'esposizione dell'equazione
di Dirac dà all'insegnante l'illusione di
avere detto « tutto » sull'atomo di idrogeno. È anche vero però che l'equazione
di Dirac è un'equazione di teoria dei
campi, e non si insegna, onestamente.

senza concetti di teoria dei campi.
Il desiderio di usare direttamente e senza
giustificazioni l'equazione di Dirac per
descrivere il moto di un singolo elettrone
nell'atomo di idrogeno deve venire sacri-
ficato ad una maggiore coerenza logica.
Il punto di vista che gli autori adottano
di escludere del tutto problemi relativi-
stici è, per lo meno, coerente. Eviden-
temente questo non permette certo di
evitare il grave problema di come e dove
insegnare la meccanica quantistica rela-
tivistica. A nostro avviso la meccanica
quantistica relativistica, o teoria dei
campi, deve venire insegnata nei nostri
corsi universitari. Sappiamo tutti che
non è possibile a tutt'oggi un'esposizione
soddisfacente e coerente della meccanica
quantistica relativistica. Tuttavia, sia
pure in questo stadio incompleto e fram-
mentario, riteniamo preferibile l'esposi-
zione della teoria del campo libero di
Dirac, e del campo di Dirac con campo
esterno classico, alla esposizione della
inesistente teoria relativistica di un sin-
golo elettrone.

R. GATTO

*Proceedings of the International Con-
ference on Nuclidic Masses,*
Mc Master University, Hamilton,
September 12-16, 1960; edited
by H. E. DUCKWORTH. Univer-
sity of Toronto Press, pp. 359
con numerose tabelle e figure.

Questi *Proceedings* sfatano ancora
una volta la leggenda secondo la quale
uno studioso che è rimasto a lavorare
nel proprio Laboratorio non può cono-
scere le ultime novità se non dopo
almeno un anno, e cioè quando si sta
già organizzando, od addirittura svol-
gendo, un secondo Congresso. Dopo la
Conferenza di Kingston, dopo quella di
Toronto e dopo quella di Gatlinburg,
anche il Congresso sulle Masse Nucleari

pubblica i suoi atti a pochi mesi di
distanza dalla sua conclusione.

A prima vista può sembrare che un
congresso sulle Masse Nucleari debba
interessare un gruppo molto limitato di
ricercatori specializzati in un ben deter-
minato campo. Ciò è inesatto in quanto
un gran numero di esperienze che si
prefiggono altri scopi, ed in particolare
quasi tutte indistintamente le ricerche
di fisica dei nuclei, forniscono diretta-
mente od indirettamente informazioni
sull'argomento.

Riferendoci alla fisica dei nuclei ba-
sterà ricordare le ricerche sulle reazioni
nucleari o sulla spettroscopia β nel corso
delle quali si misurano frequentemente
reazioni allo stato fondamentale, od
energie di transizione.

Lo studio delle masse nucleari inoltre,
se trattato statisticamente, può fornire,
mediante opportune formule, informa-
zioni sui raggi o sulle superfici nucleari,
e quindi essere impiegato nella risolu-
zione di problemi di stabilità nucleare
(fissione). Questi esempi giustificano la
presenza al Congresso di fisici come
K. A. BRUECKNER ed altri non diretta-
mente specializzati in masse nucleari.
A proposito di queste è bene ricordare
che su circa 1300 nuclidi conosciuti, sol-
tanto 870 masse erano determinato
all'epoca del Congresso!

La Conferenza si articola in otto ses-
sioni. La prima, a carattere introduttivo,
è una messa a punto sulla conoscenza
attuale delle masse nucleari. Essa con-
tiene esposizioni di J. MATTAUCH, A. H.
WAPSTRA, L. A. KÖNIG ed E. R. COHEN.
La seconda parte si occupa — da un
punto di vista teorico — della siste-
matica e della modellistica nucleare.
A. E. S. GREEN, R. BRANDT *et al.*,
R. FULLER e J. A. WHEELER, K. A,
BRUECKNER *et al.* e infine N. ZELDES
hanno trattato questi argomenti. Nella
terza parte J. B. MARION, C. P. BROWNE
et al., F. ASARO e K. N. GELLER hanno
affrontato il problema della calibrazione
in energia di macchine acceleratrici e

di sorgenti standard di calibrazione. Nella quarta parte sono state discusse le reazioni nucleari impiegate per la misura di Q. L'argomento è stato trattato da W. W. Buechner, M. Mazari et al., T. W. Bonner e da T. H. Motz. Nella quinta sessione L. Lidofsky, J. W. Knowles, R. L. Graham trattano i problemi riguardanti la determinazione dell'energia totale nel decadimento β mediante misure su elettroni e su raggi γ. Nelle parti sei e sette è trattato l'argomento tradizionale e cioè la determinazione di masse atomiche mediante la spettroscopia di massa. I lavori presentati riguardano sia gli apparecchi (J. Mattauch et al., C. M. Stevens et al., L. G. Smith) sia il loro impiego (A. O. C. Nier et al., N. R. Isenor et al., W. H. Johnson et al., K. T. Bainbridge et al., K. Ogata et al.). L'ultima sessione riguarda argomenti vari quali uno spettrometro di massa per l'analisi di particelle di elevata energia cinetica (H. Ewald), la determinazione assoluta del peso atomico dell'argento (W. R. Shields et al.), ecc. L'intera conferenza viene chiusa da un'ottima esposizione riassuntiva di A. H. Wapstra.

Facendo il bilancio degli interi Atti della Conferenza si nota che le novità apportate dai partecipanti in relazione alla necessità di compilare una nuova tavola di masse, non sono tali da promuovere tale iniziativa. Infatti ad eccezione del lavoro di Nier sulla determinazione di masse atomiche nella regione Gd—Au finora poco conosciuta, non ci sono altri motivi che giustifichino un riesame delle tavole. Anzi sono venute fuori delle discrepanze sull'energia di legame del deuterio e sull'energia dell'α del ^{210}Po che consigliano addirittura di intraprendere nuove misure o comunque una maggiore critica ai risultati finora ottenuti. Un elemento di notevole interesse è come al solito fornito dai lavori sulle sistematiche nucleari che continuano ad apportare chiarificazioni ed informazioni nel campo della fisica dei nuclei.

Per concludere, si può affermare che sia per gli argomenti trattati che per la fama degli autori, il libro risulta molto utile non soltanto agli esperti di masse nucleari ma anche a coloro che si occupano di fisica dei nuclei.

S. Sciuti

D. Kastler – *Introduction à l'électrodynamique quantique*. Dunod, Paris, 1961, pp. 333.

È ormai noto che la teoria dei campi relativistici si trova da alcuni anni in una situazione di crisi, e che nonostante i grandi progressi compiuti nel comprendere la natura delle difficoltà caratteristiche della teoria, resta tuttora non risolto il problema fondamentale della costruzione esplicita di una soluzione delle equazioni del moto. Recentemente alcuni risultati del cosidetto metodo assiomatico hanno permesso di affrontare direttamente il problema dell'esistenza di soluzioni per alcune teorie particolari, mentre, nell'ambito del formalismo canonico, sono stati raggiunti importanti risultati usando la teoria delle algebre astratte. Tutti questi sviluppi sono caratterizzati dall'uso di concetti matematici moderni e da un alto livello di rigore nella trattazione.

Se si considera che l'istruzione matematica ricevuta dai fisici nelle Università è prevalentemente classica, si comprende facilmente l'importanza che può avere un testo di teoria dei campi che adotti un linguaggio matematicamente moderno. Il volume di Kastler si pone in questo ordine di idee e si indirizza esplicitamente ai matematici, oltre che ai fisici, col proposito di suscitare nei primi interesse per i problemi creati dalla fisica e di fornire ai secondi una introduzione semplificata al linguaggio dei matematici. Il libro ha certamente il merito di essere il primo di questo genere. Tuttavia ci sembra difficile che

riesca ad assolvere il compito che si propone. Dopo una breve introduzione al formalismo Hamiltoniano in meccanica quantistica e in teoria dei campi, il libro inizia con due capitoli di carattere matematico dedicati il primo alla teoria degli spazi lineari e il secondo alle algebre multilineari negli spazi vettoriali. Per il fisico questi sono forse i capitoli più interessanti poichè costituiscono un'esposizione rigorosa e nello stesso tempo sufficientemente abbreviata di concetti che in genere si trovano discussi in trattati di lettura molto più difficile. Nel primo capitolo è inoltre contenuta una discussione della struttura del gruppo di Lorentz.

I capitoli III, IV, V e VI espongono la teoria della seconda quantizzazione nel caso dell'equazione di Schrödinger, dell'equazione di Klein-Gordon e del campo di Maxwell rispettivamente e, salvo l'uso della terminologia e delle notazioni introdotte nei primi due capitoli, son del tutto convenzionali. Più interessante è il capitolo VII sulla equazione di Dirac dove la teoria degli spinori viene trattata con maggiore generalità che non nei testi scritti da fisici. Questo capitolo va completato con l'appendice B del libro stesso.

I restanti nove capitoli seguono anch'essi nella sostanza le esposizioni convenzionali della teoria, specialmente i due capitoli X e XI sui campi in interazione dove l'autore si limita ad esporre il metodo di Dyson per la costruzione della matrice S e accenna solamente di sfuggita alle difficoltà di coerenza matematica. Dopo la discussione di alcuni esempi il volume si conclude con un capitolo dedicato ai diagrammi di Feynman e uno dedicato alla teoria del campo esterno.

Possiamo ora precisare la critica avanzata precedentemente. L'autore si è limitato alla traduzione in un linguaggio moderno di quegli sviluppi formali che hanno caratterizzato la teoria dei campi fino al 1955. I problemi più sostanziali proprio dal punto di vista

matematico affrontati più recentemente, non sono neppure accennati.

G. Jona-Lasinio

H. G. Van Bueren – *Imperfections in Crystals*. North-Holland Publishing Co., Amsterdam, 1960; pp. XVIII-676.

Forse un titolo più appropriato per questo libro sarebbe stato « Proprietà dei solidi cristallini difettosi ». Il libro non si limita difatti alla descrizione dei vari tipi di imperfezioni fisiche dei reticoli cristallini, ma passa in rassegna tutte le proprietà fisiche che in un modo o nell'altro sono influenzate dalla presenza di questi difetti. E' evidente che la maggior parte delle proprietà specifiche dei difetti si possono ricavare solo attraverso uno studio sistematico di proprietà fisiche diverse; spesso però lo studioso tende a concentrare il suo pensiero solo su alcune di queste proprietà. Il van Bueren si propone di dare un panorama più vasto delle proprietà dei solidi difettosi e di interessare il lettore alla grande varietà di idee e di modelli teorici ingegnosi che sono stati proposti, anche se talora essi hanno dovuto essere scartati in base all'evidenza sperimentale.

La conseguenza diretta di questa impostazione data dall'autore è la ponderosità del libro: esso è costituito di 31 capitoli, raccolti in tre parti: I) una parte generale introduttiva stringata e piuttosto incompleta per il lettore sprovvisto, che passa in rassegna le principali imperfezioni fisiche puntiformi e non puntiformi e ne enuncia la teoria sulla base della teoria classica dell'elasticità; II) una parte riguardante le imperfezioni nei metalli completa e ricca di bibliografia, che costituisce il grosso del trattato; essa riguarda una svariata serie di argomenti, dai difetti prodotti per deformazione plastica a quelli pro-

dotti per irraggiamento, dalla diffusione alle proprietà meccaniche e ai meccanismi di crescita dei cristalli; III) una parte finale riguardante le imperfezioni nei cristalli non metallici (pressochè esclusivamente cristalli ionici, germanio e silicio).

Le imperfezioni chimiche (impurezze) sono trattate solo incidentalmente quando queste possano interagire con le imperfezioni fisiche, in particolare con le dislocazioni.

La ponderosità del libro ha un grande svantaggio che è quello di renderne difficile la lettura. Il libro del VAN BUEREN va quindi considerato un trattato di consultazione e da questo punto di vista è molto utile per la ricchezza delle informazioni in esso contenute. Purtroppo però l'indice analitico non risponde alle esigenze di una facile consultazione « Divacancy », « double vacancy » e « vacancy pair » corrispondono ovviamente allo stesso concetto ma le pagine citate nell'indice analitico sono rispettivamente cinque, due ed una; purtroppo tra queste pagine non è citata quella dove si dà la formula della concentrazione delle coppie di posti vacanti in equilibrio termodinamico.

<div align="right">G. BOATO</div>

A. F. IOFFE – *Physics of Semiconductors*. Infosearch Limited, London, 1960, pp. 436.

Il presente volume è la traduzione inglese, aggiornata e riveduta dall'Autore, del testo pubblicato nel 1957 dall'eminente fisico sovietico recentemente scomparso.

Va subito detto che si tratta di un libro stupendo, che segue la via più diretta per affrontare il difficile argomento della fisica dei semiconduttori senza presupporre nel lettore nozioni che non siano quelle fornite dai corsi universitari istituzionali. La trattazione degli argomenti introduttivi (statistiche, teoria dei difetti nei solidi) è limitata all'essenziale, con chiarezza estrema e con un certo disdegno delle spesso tediose trattazioni matematiche alle quali in genere indulgono gli autori di altri testi analoghi finora pubblicati; è evidentemente un libro dedicato dalla prima all'ultima riga ai fisici sperimentali.

D'altra parte, dato l'enorme interesse tecnologico dei semiconduttori, si è ritenuto a ragione importante sviluppare grandemente la parte dedicata alle proprietà interessanti dal punto di vista applicativo. Ciò è stato fatto però con grande senso di equilibrio, fornendo gli elementi basilari per lo studio delle utilizzazioni dei semiconduttori ma senza mai scivolare nella tecnologia: un esempio di ciò è l'ultimo capitolo in cui si discutono le particolarità fisiche essenziali dei materiali semiconduttori e si dedica ampio spazio ai requisiti richiesti per i termoelementi.

Una discreta bibliografia correda ciascun capitolo, e fa in prevalenza riferimento alla letteratura sovietica; il presente volume è tuttavia, più che un libro di consultazione, un testo a cui il lettore può affidarsi con piena fiducia se desidera organizzare le proprie idee sulle proprietà generali dei semiconduttori.

La materia è suddivisa in sette capitoli. Di questi, i primi quattro sono dedicati alle generalità e servono da introduzione: esame dei meccanismi di conduzione nei solidi, fondamenti di teoria dei metalli e dei semiconduttori, trattazione quantistica delle proprietà dei semiconduttori.

Il quinto capitolo comprende circa metà del volume ed è dedicato agli aspetti più tipici della fisica dei semiconduttori: vi è sviluppata la trattazione di molti particolari che di solito trovano posto esclusivamente nei libri di tipo applicativo (proprietà superficiali, giunzioni, effetti termoelettrici e magnetoelettrici), ma ciò è fatto sempre con

impostazione prettamente fisica e con elegante semplicità.

Gli ultimi due capitoli sono rivolti all'esame dei metodi di ricerca propri dei semiconduttori e all'analisi delle proprietà di molti materiali di questo tipo.

Un cenno a parte meritano le dieci pagine di conclusioni in cui l'Autore, forte della sua superiore conoscenza dell'argomento e della sua vastissima esperienza, delinea criticamente la storia delle nostre conoscenze sui semiconduttori e indica taluni indirizzi per la ricerca futura.

E. GERMAGNOLI

T. H. NAPVI — *A study of the azimuthal and the zenithal distribution of cosmic rays at Gulmarg (Kashmir).* Muslim University, Aligarh, India.

Non lo si può chiamare un libro; si ha piuttosto l'impressione di trovarsi di fronte ad un articolo di rivista, forse troppo lungo (64 pagine), legato in un modesto volumetto. Si tratta della tesi di laurea presentata dall'autore per il suo dottorato (Ph. D.) nell'Ottobre 1956, e pubblicata solo 4 anni dopo per cause, dice l'Autore, « beyond my control ». Vi sono riportati essenzialmente i risultati di un accurato studio sistematico svolto dall'Autore sugli effetti geomagnetici direzionali della radiazione cosmica a Gulmarg, cioè a circa 24° N di latitudine geomagnetica e circa 2 800 m di altezza.

Il Capitolo I consiste in un breve ma chiaro quadro introduttivo dell'argomento in studio. Il Capitolo II nella descrizione, forse un po' troppo dettagliata, dell'apparecchiatura sperimentale usata: e cioè di un complesso di telescopi di contatori G-M in coincidenze triple, con assorbitore di 10 cm di Pb, disposti in modo da esplorare l'intero

emisfero contemporaneamente secondo le direzioni a 15°, 30°, 45° e 75° rispetto allo Zenit e con possibili variazioni di azimut di 15° in 15°. Nei Capitoli IV e V dopo una rassegna dei risultati conosciuti in precedenza, (il punto della situazione risulta naturalmente intorno agli anni 1954-56), sono riportati i risultati dell'Autore, considerati dal punto di vista della distribuzione azimutale e rispettivamente zenitale dell'intensità dei raggi cosmici. Nell'ultimo capitolo, infine, i risultati sono brevemente discussi dal punto di vista delle informazioni che da essi possono trarsi circa la natura delle particelle primarie. Il tutto è corredato da numerosi disegni, grafici e tabelle che costituiscono più della metà del libretto e potrebbero essere utile illustrazione dell'argomento: peccato che l'ordine di successione sia così poco curato e, soprattutto, la veste tipografica così povera.

A. M. CONFORTO

M. J. AITKEN — *Physics and Archaeology.* Interscience Publishers, New York and London, 1961, pp. X-181; $ 6.00.

In questo breve manuale è esposta per la prima volta una trattazione, pressochè completa, di tutti i mezzi che la fisica mette a disposizione per gli studi di archeologia e preistoria.

Il libro può essere considerato diviso in tre parti: la prima, comprendente quattro capitoli, tratta dei metodi fisici di prospezione del sottosuolo, la seconda, di tre capitoli, espone i metodi di datazione e la terza parte, l'ultimo capitolo, è una rassegna dei metodi per l'esame chimico dei materiali archeologici.

Il primo e quinto capitolo, che presentano le stesse caratteristiche, sono una introduzione rispettivamente alla prima e alla seconda parte: nel primo capitolo, infatti, vengono elencati quei

metodi fisici di prospezione del sotto-
suolo che hanno dato i risultati più
soddisfacenti per la localizzazione di
manufatti vari, tombe, resti di costru-
zione, ecc. sepolti; il quinto capitolo
enumera i metodi di datazione assoluta
sia radiochimici (metodo del radio-
carbonio in particolare) che magnetici
con brevi cenni anche su altri metodi
(dendrocronologia, metodo delle varve)
che, com'è noto, interessano il quater-
nario recente.

L'ultimo capitolo, come si è detto,
è dedicato a quei metodi analitici più
largamente usati per determinare quelle
caratteristiche di composizione chimica
di oggetti archeologici più significative
per uno studio diagnostico di essi.

Particolare riferimento è fatto ai
metodi di analisi strumentale più adatti
allo scopo (analisi spettrografica, analisi
per fluorescenza ai raggi X, analisi per
attivazione neutronica, misura dello scat-
tering di raggi β).

Fra i metodi e le apparecchiature
per la prospezione geofisica vengono in
particolare trattati i metodi per la loca-
lizzazione magnetica, secondo capitolo;
il magnetometro a precessione nucleare,

terzo capitolo, e infine i metodi per la
misura della resistività del sottosuolo
al quarto capitolo.

I capitoli sesto e settimo sono speci-
ficamente dedicati ai metodi di data-
zione col radiocarbonio e al metodo
magnetico rispettivamente.

La caratteristica comune della trat-
tazione dei capitoli precedentemente
citati (II, III, IV, VI e VII) è il modo
esauriente con cui i singoli metodi ven-
gono trattati: principi dei metodi, tec-
niche, applicazioni, inconvenienti e pos-
sibili errori vengono descritti in modo
tale da costituire un'utile guida a chi
voglia documentarsi sull'argomento. La
esposizione è molto chiara.

Altro pregio del testo è rappresen-
tato dalla bibliografia accuratamente
selezionata, di cui sono dotati i singoli
capitoli, da un indice bibliografico per
autori, e da un indice analitico.

In complesso, la lettura del manuale,
può essere di particolare utilità a stu-
diosi di archeologia e preistoria nonchè
a quei fisici che vogliano documentarsi
sul contributo che la loro disciplina offre
ad altri campi di ricerca.

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